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ON CHARACTERIZATIONS OF THE WEIBULL DISTRIBUTION BY THE INDEPENDENT PROPERTY OF RECORD VALUES

Min-Young Lee* and Eun-Hyuk Lim**

Abstract. We present characterizations of the Weibull distribution by the independent property of record values that $F(x)$ has a Weibull distribution if and only if $\frac{X_{U(m)}}{X_{U(n)}}$ and $X_{U(n)}$ or $\frac{X_{U(n)}}{X_{U(n)} \pm X_{U(m)}}$ and $X_{U(n)}$ are independent for $1 \leq m < n$.

1. Introduction

Let $\{X_n, n \geq 1\}$ be a sequence of independent identically distributed (i.i.d.) random variables with cumulative distribution function(cdf) $F(x)$ and probability density function(pdf) $f(x)$. Let $Y_n = max(min)$ $\{X_1, X_2, \cdots, X_n\}$ for $n \geq 1$. We say X_j is an upper(lower) record value of this sequence, if $Y_j > (<) Y_{j-1}$ for $j > 1$. By definition, X_1 is an upper as well as a lower record value. The indices at which the upper record values occur are given by the record times $\{U(n), n \geq 1\}$, where $U(n) = min\{j \mid j > U(n-1), X_j > X_{U(n-1)}, n \geq 2\}$ with $U(1) = 1$. We assume that all upper record values $X_{U(i)}$ for $i \geq 1$ occur at a sequence $\{X_n, n \geq 1\}$ of i.i.d. random variables.

We call the random variable $X \in WEI(\alpha)$ if the corresponding probability cdf $F(x)$ of X is of the form

$$F(x) = \begin{cases} 1 - e^{-x^{\alpha}}, & x > 0,\ \alpha > 0 \\ 0, & otherwise. \end{cases}$$

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Correspondence should be addressed to Min-Young Lee, leemy@dankook.ac.kr.

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Ahsanullah(1995) proved that $F(x) = 1 - e^{-\frac{x}{\alpha}}, x > 0, \alpha > 0$, if and only if $X_{U(n)} - X_{U(m)}$ and $X_{U(m)}$, $0 < m < n$ are independent. Also, Lee and Chang(2008) obtained characterization that $F(x) = 1 - e^{-x^{\alpha}}$ for all $x > 0$ and $\alpha > 0$, if and only if $\frac{X_{U(n+1)}}{X_{U(n+1)}+X_{U(n)}}$ and $X_{U(n+1)}$ are independent for $n \geq 1$.

In this paper we generalize the results of Lee and Chang(2008) and obtain characterizations of the Weibull distribution by the independent property of record values.

2. Main results

THEOREM 2.1. *Let $\{X_n, n \geq 1\}$ be a sequence of i.i.d. random variables with cdf $F(x)$ which is absolutely continuous with pdf $f(x)$ and $F(0) = 0$ and $F(x) < 1$ for all $x > 0$. Then $F(x) = 1 - e^{-x^{\alpha}}$ for all $x > 0$, $\alpha > 0$, if and only if $\frac{X_{U(m)}}{X_{U(n)}}$ and $X_{U(n)}$ are independent for $1 \leq m < n$.*

Proof. If $F(x) = 1 - e^{-x^{\alpha}}$ for all $x > 0$, $\alpha > 0$, then the joint pdf $f_{m,n}(x, y)$ of $X_{U(m)}$ and $X_{U(n)}$ is found to be

$$f_{m,n}(x,y) = \frac{\alpha^2}{\Gamma(m)\Gamma(n-m)} x^{\alpha m - 1}\{y^{\alpha} - x^{\alpha}\}^{n-m-1} y^{\alpha - 1} e^{-y^{\alpha}}$$

for all $1 \leq m < n$, $\alpha > 0$.

Consider the functions $V = \frac{X}{Y}$ and $W = Y$. It follows that $x = vw$, $y = w$ and the absolute value of Jacobian of the transformation is $\mid J \mid = w$. Thus we can write the joint pdf $f_{V,W}(v, w)$ of V and W as

$$(2.1) \quad f_{V,W}(v,w) = \frac{\alpha^2}{\Gamma(m)\Gamma(n-m)} v^{\alpha m - 1}(1 - v^{\alpha})^{n-m-1} w^{\alpha n - 1} e^{-w^{\alpha}}$$

for all $0 < v < 1$, $w > 0$ and $\alpha > 0$.

The marginal pdf of V is given by

$$(2.2) \quad \begin{aligned} f_V(v) &= \int_0^{\infty} f_{V,W}(v,w)dw \\ &= \frac{\alpha\Gamma(n)}{\Gamma(m)\Gamma(n-m)} v^{\alpha m - 1}(1 - v^{\alpha})^{n-m-1}, \end{aligned}$$

for all $1 \leq m < n$, $\alpha > 0$.

Also, the pdf $f_W(w)$ of W is given by

$$f_W(w) = \frac{R(w)^{n-1}}{\Gamma(n)} f(w)$$
$$(2.3) \qquad = \frac{\alpha w^{\alpha n-1}}{\Gamma(n)} e^{-w^{\alpha}}, \; where\ R(w) = -ln(1-F(w)).$$

From (2.1), (2.2) and (2.3), we obtain $f_{V,W}(v,w) = f_V(v) f_W(w)$.
Hence V and W are independent for $1 \leq m < n$.

Now we will prove the sufficient condition. Let us use the transformation $V = \frac{X_{U(m)}}{X_{U(n)}}$ and $W = X_{U(n)}$. It follows that $\mid J \mid = w$. Thus we can write the joint pdf $f_{V,W}(v,w)$ of V and W as

$$(2.4) \qquad f_{V,W}(v,w) = \frac{R(vw)^{m-1}}{\Gamma(m)} r(vw) \frac{\{R(w) - R(vw)\}^{n-m-1}}{\Gamma(n-m)} w f(w)$$

for all $0 < v < 1$, $w > 0$ and $\alpha > 0$, where $r(x) = \frac{d}{dx}(R(x))$.

The pdf $f_W(w)$ of W is given by

$$(2.5) \qquad f_W(w) = \frac{R(w)^{n-1}}{\Gamma(n)} f(w)$$

for $n \geq 2$.

We get the pdf $f_V(v)$ of V from (2.4) and (2.5) as

$$f_V(v) = \frac{\Gamma(n)}{\Gamma(m)\Gamma(n-m)} \frac{R(vw)^{m-1}}{R(w)^{n-1}} r(vw) w (R(w) - R(vw))^{n-m-1}$$
$$= \frac{\Gamma(n)}{\Gamma(m)\Gamma(n-m)} \left(1 - \frac{R(vw)}{R(w)}\right)^{n-m-1} \left(\frac{R(vw)}{R(w)}\right)^{m-1} \left(\frac{\partial}{\partial v} \frac{R(vw)}{R(w)}\right).$$

By the independent property of V and W, the pdf $f_V(v)$ of V is a function of v only [see Ahsanullah(1995), p.48] . Thus we must have

$$(2.6) \qquad R(vw) = R(v)R(w).$$

By functional equations[see aczel(1996)], the only continuous solution of (2.6) with the boundary condition $R(0) = 0$ is

$$R(x) = x^{\alpha}$$

for all $x > 0$ and $\alpha > 0$. Thus we have $F(x) = 1 - e^{-x^{\alpha}}$.

This completes the proof. □

THEOREM 2.2. *Let $\{X_n, n \geq 1\}$ be a sequence of i.i.d. random variables with cdf $F(x)$ which is absolutely continuous with pdf $f(x)$ and $F(0) = 0$ and $F(x) < 1$ for all $x > 0$. Then $F(x) = 1 - e^{-x^{\alpha}}$ for all*

$x > 0$ and $\alpha > 0$, if and only if $\frac{X_{U(n)}}{X_{U(n)}+X_{U(m)}}$ and $X_{U(n)}$ are independent for $1 \le m < n$.

Proof. The joint pdf $f_{m,n}(x,y)$ of $X_{U(m)}$ and $X_{U(n)}$ is

$$f_{m,n}(x,y) = \frac{R(x)^{m-1}}{\Gamma(m)} r(x) \frac{\{R(y)-R(x)\}^{n-m-1}}{\Gamma(n-m)} f(y)$$

for all $1 \le m < n$, $\alpha > 0$, where $R(x) = -ln(1-F(x))$ and $r(x) = \frac{d}{dx}(R(x))$.

Consider the functions $V = \frac{X_{U(n)}}{X_{U(n)}+X_{U(m)}}$ and $W = X_{U(n)}$. It follows that $x_{U(m)} = \frac{w(1-v)}{v}$, $x_{U(n)} = w$ and $\mid J \mid = \frac{w}{v^2}$. Thus we can write the joint pdf $f_{V,W}(v,w)$ of V and W as

$$f_{V,W}(v,w) = \frac{R(\frac{w(1-v)}{v})^{m-1}}{\Gamma(m)} r(\frac{w(1-v)}{v}) \frac{\{R(w)-R(\frac{w(1-v)}{v})\}^{n-m-1}}{\Gamma(n-m)} \frac{w}{v^2} f(w).$$

for all $\frac{1}{2} < v < 1$, $w > 0$ and $\alpha > 0$.

If $F(x) = 1 - e^{-x^\alpha}$ for all $x > 0$, $\alpha > 0$, then we get

$$\begin{aligned} f_{V,W}(v,w) &= \frac{\alpha^2}{v^2\Gamma(m)\Gamma(n-m)} \left(\frac{1-v}{v}\right)^{\alpha m-1} \\ &\times \left(1 - \left(\frac{1-v}{v}\right)^\alpha\right)^{n-m-1} w^{\alpha n-1} e^{-w^\alpha}. \end{aligned} \tag{2.7}$$

The marginal pdf of V is given by

$$\begin{aligned} f_V(v) &= \int_0^\infty f_{V,W}(v,w)dw \\ &= \frac{\alpha\Gamma(n)}{v^2\Gamma(m)\Gamma(n-m)} \left(\frac{1-v}{v}\right)^{\alpha m-1} \left(1 - \left(\frac{1-v}{v}\right)^\alpha\right)^{n-m-1}, \end{aligned} \tag{2.8}$$

for all $1 \le m < n$, $\alpha > 0$.

Also, the pdf $f_W(w)$ of W is given by

$$f_W(w) = \frac{R(w)^{n-1}}{\Gamma(n)} f(w) = \frac{\alpha w^{\alpha n-1}}{\Gamma(n)} e^{-w^\alpha}. \tag{2.9}$$

From (2.7), (2.8) and (2.9), we obtain $f_{V,W}(v,w) = f_V(v) f_W(w)$. Hence V and W are independent for $1 \le m < n$.

Now we will prove the sufficient condition. Let us use the transformation $V = \frac{X_{U(n)}}{X_{U(n)}+X_{U(m)}}$ and $W = X_{U(n)}$. The Jacobian of the transformation is $\mid J \mid = \frac{w}{v^2}$. Thus we can write the joint pdf $f_{V,W}(v,w)$ of V and W as

$$\begin{aligned}f_{V,W}(v,w) &= \frac{R(\frac{w(1-v)}{v})^{m-1}}{\Gamma(m)} r(\frac{w(1-v)}{v}) \\ &\quad \times \frac{\{R(w) - R(\frac{w(1-v)}{v})\}^{n-m-1}}{\Gamma(n-m)} \frac{w}{v^2} f(w)\end{aligned} \tag{2.10}$$

for all $\frac{1}{2} < v < 1$, $w > 0$ and $\alpha > 0$.

The pdf $f_W(w)$ of W is given by

$$f_W(w) = \frac{R(w)^{n-1}}{\Gamma(n)} f(w) \tag{2.11}$$

for $n \geq 2$.

Since V and W are independent, we get the pdf $f_V(v)$ of V from (2.10) and (2.11) as

$$\begin{aligned}f_V(v) &= \frac{\Gamma(n)}{\Gamma(m)\Gamma(n-m)} \frac{R(\frac{w(1-v)}{v})^{m-1}}{R(w)^{n-1}} r(\frac{w(1-v)}{v}) \frac{w}{v^2} \\ &\quad \times \left(R(w) - R(\frac{w(1-v)}{v}) \right)^{n-m-1} \\ &= \frac{\Gamma(n)}{\Gamma(m)\Gamma(n-m)} \left(1 - \frac{R(\frac{w(1-v)}{v})}{R(w)} \right)^{n-m-1} \\ &\quad \times \left(\frac{R(\frac{w(1-v)}{v})}{R(w)} \right)^{m-1} \left(-\frac{\partial}{\partial v} \frac{R(\frac{w(1-v)}{v})}{R(w)} \right).\end{aligned}$$

By the independent property of V and W, the pdf $f_V(v)$ of V is a function of v only. Thus we must have

$$R\left(\frac{w(1-v)}{v}\right) = R(w) R\left(\frac{(1-v)}{v}\right). \tag{2.12}$$

By functional equations[see aczel(1996)], the only continuous solution of (2.12) with the boundary condition $R(0) = 0$ is

$$R(x) = x^{\alpha}$$

for all $x > 0$ and $\alpha > 0$. Thus we have $F(x) = 1 - e^{-x^{\alpha}}$.

This completes the proof. □

THEOREM 2.3. *Let $\{X_n, n \geq 1\}$ be a sequence of i.i.d. random variables with common distribution function $F(x)$ which is absolutely continuous with pdf $f(x)$ and $F(0) = 0$ and $F(x) < 1$ for all $x > 0$. Then*

$F(x) = 1 - e^{-x^\alpha}$ for all $x > 0$ and $\alpha > 0$, if and only if $\frac{X_{U(n)}}{X_{U(n)} - X_{U(m)}}$ and $X_{U(n)}$ are independent for $1 \leq m < n$.

Proof. We can prove it by the same way as in Theorem 2.2. □

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Department of Mathematics
Dankook University
Cheonan 330-714, Republic of Korea
E-mail: leemy@dankook.ac.kr

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Department of Mathematics
Dankook University
Cheonan 330-714, Republic of Korea
E-mail: ehlim@dankook.ac.kr