

LOGARITHMIC CAPACITY UNDER CONFORMAL MAPPINGS OF THE UNIT DISC

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ABSTRACT. If $P(f, r)$ is the set of endpoints of radii which have length greater than or equal to $r > 0$ under a conformal mapping f of the unit disc. Then for large r , the logarithmic capacity of $P(f, r)$, $\frac{1}{2\sqrt{r}} \leq \text{cap}(P(f, r)) \leq \frac{k}{\sqrt{r}}$. Where k is the positive constant.

1. Modulus and logarithmic capacity

The theory of modulus has been successfully applied to analytic functions of a complex variable, and it has found application in the study of conformal mappings.

Throughout this note, $\Omega = \{z\}$ will denote the complex plane, D is a domain in Ω . And U is the unit disc in Ω . A curve $\gamma : I \rightarrow \Omega$ is a continuous mapping of an interval I . If we speak of a curve in D , then we allow the endpoints of the curve to lie on ∂D . A curve γ in D connects two sets $A, B \subseteq \bar{D}$, if γ has one endpoint in A and one in B . We denote by $\text{len}(\gamma)$ the euclidean length of γ .

DEFINITION 1.1. ([1]) The modulus $\text{mod}(\Gamma)$ of a family Γ of locally rectifiable curves (simply, curves or arcs) in a domain D is defined as

$$\text{mod}(\Gamma) = \inf_{\rho} \int \int_D \rho(z)^2 dm_2(z).$$

Where m_2 is two-dimensional Lebesgue measure and infimum is taken over all non-negative Borel measurable functions ρ that satisfy

$$\int_{\gamma} \rho(z) |dz| \geq 1$$

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for all $\gamma \in \Gamma$, where $|dz|$ means integration with respect to euclidean arc-length. We shall call a function $\rho(z)$ in D *admissible* in association with Γ . Obviously $0 \leq \text{mod}(\Gamma) \leq \infty$.

PROPOSITION 1.2. ([9]) If $\Gamma_1 = \{\gamma\}$ is a curve family in some domain D_1 , $f : D_1 \rightarrow D_2$ is a conformal mapping and Γ_2 is the curve family in D_2 consisting of the curves $f \circ \gamma$, then

$$\text{mod}(\Gamma_1) = \text{mod}(\Gamma_2).$$

EXAMPLE 1.3. Let T be a Jordan domain with three distinguished boundary points and Γ the family of all curves in T which touch all three sides, then

$$\text{mod}(\Gamma) = \frac{1}{\sqrt{3}}.$$

In fact, by Proposition 1.2, we begin by mapping conformally on an equilateral triangle with side 1. The minimum length of $\gamma \in \Gamma$ is that of the altitude: $\sqrt{3}/2$. We set $\rho = 2/\sqrt{3}$. Then ρ is admissible in association with Γ and it follows that $\text{mod}(\Gamma) = (2/\sqrt{3})^2(\sqrt{3}/4) = 1/\sqrt{3}$.

EXAMPLE 1.4. Let R be a rectangle of sides a and b , Γ the family of all curves in R which join the two sides of length a . Then

$$\text{mod}(\Gamma) = \frac{a}{b}.$$

In fact, since the minimum length of $\gamma \in \Gamma$ is b , we set $\rho = 1/b$. Then ρ is admissible in association with Γ , and we obtain $\text{mod}(\Gamma) = a/b$.

DEFINITION 1.5. ([2]) Let E be a bounded Borel set in Ω , μ a positive mass-distribution on E with total mass unity. Then

$$\mathbb{U}^\mu(z) = \int_E \log \left| \frac{1}{z - \zeta} \right| d\mu(\zeta)$$

is called a *logarithmic potential* of μ on E , where

$$\mathbb{V}_\mu(E) = \sup_{z \in E} \mathbb{U}^\mu(z), \quad \mathbb{V} = \inf_{\mu} \mathbb{V}_\mu(E).$$

We define the *logarithmic capacity* (simply, *capacity*), $\text{cap}(E)$ of E by

$$\text{cap}(E) = \exp(-\mathbb{V}).$$

Obviously $0 \leq \text{cap}(E) < \infty$.

EXAMPLE 1.6. For the Cantor ternary set $E\{2/3\}$, $\text{cap}(E) \geq 1/18$.

PROPOSITION 1.7. ([10]) The capacity of a countable set is also zero, and the union of a countable set of sets of capacity zero is of capacity zero.

The following statements which relates the modulus and capacity is needed in the proof of the theorem.

EXAMPLE 1.8. ([10]) Let E be a compact set in U and Γ be the family of all curves which join $\{z \mid |z| = 1\}$ to E . Then

$$\text{mod}(\Gamma) = 0 \quad \text{if and only if} \quad \text{cap}(E) = 0.$$

THEOREM 1.9. ([9]) Let E be a Borel subset of ∂U and $\Gamma(E, \alpha)$ the family of all curves γ in $D = \{z \in U \mid \alpha < |z| < 1\}$ that connect $\{z \mid |z| = \alpha\}$ and E .

$$\text{cap}(E) \leq \frac{1 + \alpha}{\sqrt{\alpha}} (-1)^{\frac{1}{\Gamma(E, \alpha)}},$$

where $\alpha > 0$ is a sufficiently small constant.

2. Capacity under some conformal mappings

Now we are ready to state our result. The following theorem states that if $f : U \rightarrow \Omega$ is conformal, then the set of radii whose images under f have infinite length has vanishing capacity.

THEOREM 2.1. Let $f : U \rightarrow \Omega$ be a conformal mapping with $f'(0) = 1$. If $P(f, r)$ is the set of all $p \in \partial U$ with

$$\text{len}(f([0, p])) \geq r > 0,$$

then

$$\text{cap}(P(f, r)) \leq \frac{k}{\sqrt{r}},$$

where k is the positive constant. And for large r , there exist functions f , such that

$$\text{cap}(P(f, r)) \geq \frac{1}{2\sqrt{r}}.$$

For our proof of the theorem 2.1, we will need the followings.

THEOREM 2.2. ([9]) Let $f : U \rightarrow \Omega$ be a conformal mapping, γ a curve in U with endpoints $0, p \in \partial U$ and $[0, p)$ the radius of U with endpoint p . Then

$$\text{len}(f([0, p))) \leq c \text{len}(f \circ \gamma),$$

where c is a positive constant.

The following lemma states a modulus estimate. It shows us the usefulness of the method of modulus.

LEMMA 2.3. *Let D be a domain and Γ a family of curves in D which have one endpoint in a compact set $F \subseteq \bar{D}$. Suppose F is contained in a disc of diameter $\eta > 0$ centered at the origin. If $L \geq \eta$ and $\text{len}(\gamma) \geq L$ for all $\gamma \in \Gamma$, then*

$$\text{mod}(\Gamma) \leq \frac{2\pi}{\log(1 + L/\eta)}.$$

Proof. In addition to our assumptions on M we may assume that there exists at least one rectifiable curve in D which connects a point in D to a point in M . For otherwise it is easy to see that

$$\text{mod}(\Gamma) = 0.$$

(Consider test functions ρ which are equal to $\alpha > 0$ on $B \cap D$ where B is some open disc containing M and 0 elsewhere. Let α tend to 0.)

For $w \in D$ define

$$l(w) = \inf_{\gamma} \text{len}(\gamma),$$

where the infimum is taken over all curves in D connecting w and M . The additional assumption on M implies that

$$l(w) < \infty$$

for all $w \in D$. The function l is continuous on D and satisfies

$$l(w) \geq |w| - \frac{\eta}{2}$$

for $w \in D$. Moreover, if $\gamma : [0, t_0] \rightarrow \Omega$ is a curve in D parameterized with respect to arc-length and if $\gamma(0) \in M$, then

$$l(\gamma(t)) \leq t$$

for $t \in (0, t_0]$.

Define $\rho : D \rightarrow [0, \infty)$ by

$$\rho(w) = \begin{cases} \frac{1}{(\log(1+L/\eta))(\eta+l(w))} & \text{if } l(w) \leq L, \\ 0 & \text{otherwise.} \end{cases}$$

Obviously, the function ρ is Borel measurable and we claim that

$$\int_{\gamma} \rho(w) |dw| \geq 1$$

for all $\gamma \in \Gamma$. Hence forth, $\log(1 + L/\eta)$ will simply denote δ .

To see this let $\gamma \in \Gamma$ be arbitrary. We may assume that $\gamma : I \rightarrow \Omega$ has an arc-length parametrization with $I = [0, \text{len}(\gamma)]$ and that $\gamma(0) \in M$. We have $l(\gamma(s)) \leq s$ for all $s \in I - \{0\}$. By assumption $\text{len}(\gamma) \geq L$ and so

$$\begin{aligned} \int_{\gamma} \rho(w) |dw| &\geq \frac{1}{\delta} \int_0^L \frac{ds}{\eta + l(\gamma(s))} \\ &\geq \frac{1}{\delta} \int_0^L \frac{ds}{\eta + s} \\ &= 1. \end{aligned}$$

Therefore, if $L \geq \eta$

$$\begin{aligned} \text{mod}(\Gamma) &\leq \int \int_D \rho(w)^2 dm_2(w) \\ &= \frac{1}{\delta^2} \int \int_{\{w \in D : l(w) \leq L\}} \frac{dm_2(w)}{(\eta + l(w))^2} \\ &\leq \frac{1}{\delta^2} \int \int_{\{w \in \Omega : |w| \leq L + \eta/2\}} \frac{dm_2(w)}{(\eta/2 + |w|)^2} \\ &= \frac{2\pi}{\delta} + 2\pi \frac{\log 2 - 1 + \eta/(2L + 2\eta)}{\delta^2} \\ &\leq \frac{2\pi}{\delta} \\ &= \frac{2\pi}{\log(1 + L/\eta)}. \end{aligned}$$

This completes the proof of the lemma. $\square$

3. Proof of the theorem 2.1

The idea of the proof is essentially the same as in [9]. A limiting argument is employed in Pfluger's theorem which is related to the concept of reduced extremal distance ([1], [8]). The new ingredients in our proof are the more refined modulus estimate of the lemma and the use of the Gehring-Hayman theorem. The proof will show that for the constant k in the theorem we can take

$$k = \sqrt{2c}$$

where c is the constant in the Gehring-Hayman theorem ([7]).

We use the notation of the theorem and may assume $f(0) = 0$. Let $\alpha \in (0, 1)$ be arbitrary. Let $\Gamma_1(\alpha)$ be the family of all curves in $\{z \in U \mid \alpha < |z| < 1\}$ connecting $\{z \in U \mid |z| = \alpha\}$ and $P(f, r)$. We leave it to the reader to show that the set $P(f, r)$ is a countable intersection of open subsets of ∂U . Hence it is a Borel set.

Suppose $\gamma \in \Gamma_1(\alpha)$ and let $z_0 \in U$, $|z_0| = \alpha$, and $p \in P(f, r)$ be the endpoints of γ . Let $[0, z_0]$ be the line segment with endpoints 0 and z_0 . If we join $[0, z_0]$ and γ , then we get a curve $\tilde{\gamma}$ in U connecting 0 and p . By the Gehring-Hayman theorem and by definition of $P(f, r)$

$$\begin{aligned} \text{len}(f \circ \tilde{\gamma}) &\geq \frac{1}{c} \text{len}(f([0, p])) \\ &\geq \frac{r}{c}. \end{aligned}$$

By Koebe's distortion theorem([9]),

$$|f'(z)| \leq 1 + 5\alpha$$

if $|z| \leq \alpha$ and $\alpha > 0$ is sufficiently small. It follows that for small α

$$\begin{aligned} \text{len}(f \circ \gamma) &\geq \frac{r}{c} - (\alpha + 5\alpha^2) \\ &= L. \end{aligned}$$

We now apply the lemma for the region

$$D = f(U - \{z \in U \mid |z| \leq \alpha\}),$$

the compact set

$$M = f(\{z \in U \mid |z| = \alpha\}) \subseteq \bar{D}$$

and the curve family

$$\Gamma_2(\alpha) = \{f \circ \gamma \mid \gamma \in \Gamma_1(\alpha)\}.$$

By Koebe's distortion theorem M is contained in a disc centered at the origin of diameter

$$\eta = 2\alpha(1 + 3\alpha)$$

for small $\alpha > 0$. It follows that for small $\alpha > 0$

$$\begin{aligned} \text{mod}(\Gamma_1(\alpha)) &= \text{mod}(\Gamma_2(\alpha)) \\ &\leq \frac{2\pi}{\log \left(\frac{r/c + \alpha + \alpha^2}{2\alpha(1+3\alpha)} \right)}. \end{aligned}$$

Hence Pfluger's theorem implies

$$\begin{aligned}
\operatorname{cap}(P(f, r)) &\leq \liminf_{\alpha \rightarrow 0} \frac{(1 + \alpha)\sqrt{2 + 6\alpha}}{\sqrt{r/c + \alpha + \alpha^2}} \\
&= \frac{\sqrt{2c}}{\sqrt{r}} \\
&= \frac{k}{\sqrt{r}}.
\end{aligned}$$

The first part of the theorem follows.

For the second part consider the Koebe function

$$f(z) = \frac{z}{(1 - z)^2}, \quad z \in \Omega - \{1\}.$$

If $r > \frac{1}{4}$ there exists $\varphi \in (0, \pi)$ such that

$$R = \frac{1}{4 \sin^2(\varphi/2)}.$$

Since

$$\operatorname{len}(f([0, p])) \geq |f(p)|$$

for $p \in \partial U$, we have

$$A = \{e^{i\beta} \mid \beta \in [-\varphi, \varphi]\} \subseteq P(f, r).$$

Since the capacity of the circular arc A is

$$\operatorname{cap}(A) = \sin \frac{\varphi}{2}$$

([9]) we obtain

$$\operatorname{cap}(P(f, r)) \geq \frac{1}{2\sqrt{r}}.$$

This completes the proof of the theorem.

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