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DOMAINS WITH C^k CR CONTRACTIONS

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Abstract. Let Ω be a domain with smooth boundary in $\mathbb{C}^{n+1}$ and let $p \in \partial\Omega$. Suppose that Ω is Kobayashi hyperbolic and p is of Catlin multi-type $\tau = (\tau_0, \ldots, \tau_n)$. In this paper, we show that Ω admits a C^k contraction at p with $k \geq |\tau| + 1$ if and only if Ω is biholomorphically equivalent to a domain defined by a weighted homogeneous polynomial.

1. Introduction

Let Ω be a domain in $\mathbb{C}^{n+1}$ with smooth boundary and let $p \in \partial\Omega$. We call p an *orbit accumulation boundary point* if there exist a point $q \in \Omega$ and a family of biholomorphic self maps $f_j \in \text{Aut}(\Omega)$ such that $\lim_{j\to\infty} f_j(q) = p$. A map $f \in \text{Aut}(\Omega)$ is called a *contraction* at $p \in \partial\Omega$ if f extends up to $\partial\Omega$ as a C^1 map such that $f(p) = p$ and $\|df_p\| < 1$. If Ω admits a contraction at p, then p is an orbit accumulation point.

In [7], Kim and Yoccoz proved that if Ω is a smoothly bounded domain admitting a contraction f at $p \in \partial\Omega$ which is C^∞ up to $\partial\Omega$ near p, then Ω is biholomorphic to a domain defined by a weighted homogeneous polynomial. In this paper we prove that the analogous result of Kim and Yoccoz also holds under less regularity assumption on f. More precisely, we prove the following:

Theorem 1.1. *Let Ω be a Kobayashi hyperbolic domain in $\mathbb{C}^{n+1}$ with C^∞ boundary and let $p \in \partial\Omega$ with Catlin multi-type $\tau = (\tau_0, \ldots, \tau_n)$. Suppose Ω admits an automorphism $f \in \text{Aut}(\Omega) \cap C^k(\overline{\Omega})$ contracting at $p \in \partial\Omega$ for some $k \geq |\tau| + 1$, then Ω is biholomorphic to a domain defined by a weighted homogeneous polynomial.*

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Note that a bounded domain is Kobayashi hyperbolic.

Since a domain defined by a weighted homogeneous polynomial admits dilation, as a corollary, we prove the following:

Corollary 1.2. *Let* Ω *and* $p \in \partial\Omega$ *be as in Theorem 1.1. Then there exists a noncompact one-parameter family of automorphisms* $\{f_t\}_{t\in\mathbb{R}}$ *such that* $\lim_{t\to\infty} f_t(q) = p$ *for some* $q \in \Omega$.

In §1, we review the concept of multi-type introduced by Catlin and then define Catlin model. Then we prove basic properties of model. In §2, we construct an infinitesimal CR automorphism of M which is preserved by f up to a scaling factor. In §3, we prove a partial linearization for C^k contractions. Finally in §4, we prove our results.

2. Preliminaries

In this section we introduce Catlin's multi-type, weight and weighted homogeneous models. We refer to [6] for details.

Let M be a germ of C^∞ real hypersurface in $\mathbb{C}^{n+1}$ passing through the origin. Take a C^∞ defining function ρ of M such that $d\rho|_M \neq 0$. Let $\tau = (\tau_0, \dots, \tau_n)$ be an $(n+1)$-tuple of integers satisfying:

(i) $\tau_0 \leq \cdots \leq \tau_n$.

(ii) For every $(n+1)$-tuple of non-negative integers $\alpha = (\alpha_0, \dots, \alpha_n)$ and $\beta = (\beta_0, \dots, \beta_n)$ it holds that

$$\left.\frac{\partial^{|\alpha|+|\beta|}\rho}{\partial z_0^{\alpha_0}\cdots\partial z_n^{\alpha_n}\partial \bar{z}_0^{\beta_0}\cdots\partial \bar{z}_n^{\beta_n}}\right|_0 = 0, \tag{2.1}$$

whenever $\displaystyle\sum_{i=0}^{n} \frac{\alpha_i + \beta_i}{\tau_i} < 1$.

Exploit the lexicographic order on the multi-indices. Then one may consider the maximum $(n+1)$-tuple τ among all possible $(n+1)$-tuples satisfying (i) and (ii) above. Call this τ a *distinguished weight* for M at 0 with respect to the standard coordinate system. *Catlin's multi-type* of M at 0 is defined to be the supremum of distinguished weights with respect to the lexicographic ordering, where the supremum is taken over all possible defining functions and holomorphic local coordinates at 0. Let $\tau = (\tau_0, \dots, \tau_n)$ denote Catlin's multi-type of M at 0. Since $d\rho|_M \neq 0$, it is obvious that $\tau_0 = 1$. Furthermore, τ_1 is the Kohn-Bloom-Graham type of M at 0.([4])

Now let $\tau = (\tau_0, \ldots, \tau_n)$ be Catlin's multi-type of M at 0. Define

$$|\tau| = \tau_0 + \cdots + \tau_n.$$

In this section we assume that $|\tau| < \infty$. Then we can choose a holomorphic coordinate system and a defining function ρ of M realizing Catlin multi-type.([6]) Fix such a defining function ρ and a local coordinate system $(z_0, z) = (z_0, \ldots, z_n)$ that realize Catlin's multi-type. We assign weights $m_0, \ldots, m_n$ to $z_0, \ldots, z_n$, respectively, by

$$m_0 = 1,\ m_j = \frac{1}{\tau_j},\ j = 1, \ldots, n.$$

By [4], in a suitable open neighborhood, say U of the origin in $\mathbb{C}^{n+1}$, M is defined by

$$\rho = \text{Im } w - P(z, \overline{z}) + r(z, \overline{z}, \text{Re } w),$$

where $w = z_0$ and $P(z, \overline{z})$ is a weighted homogeneous polynomial in $z_1, \ldots, z_n$, i.e.,

$$P(\lambda^{m_1} z_1, \ldots \lambda^{m_n} z_n, \lambda^{m_1} \bar{z}_1, \ldots \lambda^{m_n} \bar{z}_n) = \lambda P(z, \overline{z})$$

without any pluri-harmonic terms and $r(z, \bar{z}, \text{Re } w)$ is a C^∞ function whose formal power series consists of terms of higher weights.

Definition 2.1. A real hypersurface defined by

$$\text{Im } w = P(z, \bar{z})$$

is called a *Catlin model* for M.

We now decompose the holomorphic tangent space $T_0^{1,0}M$ as follows:

Choose integers $\ell_1, \ell_2, \ldots, \ell_s$ such that $1 = \ell_1 < \ell_2 < \ldots < \ell_s \leq n$ and

$$\begin{aligned} \tau_1 \;&=\; \ldots = \tau_{\ell_2 - 1} \\ &< \; \tau_{\ell_2} = \ldots = \tau_{\ell_3 - 1} \\ &\quad \vdots \\ &< \; \tau_{\ell_s} = \ldots = \tau_n. \end{aligned}$$

Then let

$$\begin{aligned}\mathcal{W}_0 &= \text{Span}\left\{\frac{\partial}{\partial z_0}\right\}\\ \mathcal{W}_1 &= \text{Span}\left\{\frac{\partial}{\partial z_j} \ : \ \ell_1 \le j < \ell_2\right\}\\ &\vdots\\ \mathcal{W}_s &= \text{Span}\left\{\frac{\partial}{\partial z_i} \ : \ \ell_s \le j \le n\right\}\end{aligned}$$

We shall also denote by $\mathcal{V}_t = \mathcal{W}_t \oplus \cdots \oplus \mathcal{W}_s$ for each $t = 0, 1, \ldots, s$. Write

$$x = (z_0, z_1, \ldots, z_n) = (w; Z_1; \ldots; Z_s),$$

where

$$Z_t = (z_{\ell_t}, \ldots, z_{\ell_{t+1}-1})$$

and write

$$z = (Z_1, \ldots, Z_s).$$

It is possible to express $P(z, \bar{z})$ as

$$P(z,\bar{z}) = P_1(Z_1, \bar{Z}_1) + P_2(Z_1, Z_2, \bar{Z}_1, \bar{Z}_2) + \ldots + P_s(Z_1, \ldots, Z_s, \bar{Z}_1, \ldots, \bar{Z}_s)$$

where each P_t is a polynomial satisfying

$$P_t|_{\{Z_t=0\}} \equiv 0.$$

Notice that each P_t is a weighted homogeneous polynomial in such a way that

$$P_t(\lambda^{m_1} z_1, \ldots \lambda^{m_{\ell_{t+1}-1}} z_{\ell_{t+1}-1}, \lambda^{m_1} \bar{z}_1, \ldots \lambda^{m_{\ell_{t+1}-1}} \bar{z}_{\ell_{t+1}-1}) = \lambda P_t(z, \bar{z}).$$

For each $\varepsilon > 0$, define a map $S_\varepsilon : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ by

$$S_\varepsilon(w, z) = (\varepsilon w, \varepsilon^{m_1} z_1, \ldots, \varepsilon^{m_n} z_n).$$

Definition 2.2. A polynomial G is said to have *weight* ω if

$$G \circ S_\varepsilon(w, z) = \varepsilon^\omega \widetilde{G}(w, z) + o(\varepsilon^\omega)$$

for some non-zero polynomial $\widetilde{G}$. The zero polynomial is understood as having *weight* ∞.

As in §2 of [6], we can show that a CR diffeomorphism $f : (M, 0) \to (M, 0)$ can be written as

$$f = (\mu w, Lz) + h(w, z) + o(|(w, z)|^k),$$

where $\mu \in \mathbb{R}$, Lz is a complex linear map in z, $h(w, z)$ is a polynomial in w, z of degree k such that $S_\varepsilon^{-1} \circ h \circ S_\varepsilon \in O(1)$ as $\varepsilon \to 0$. Furthermore, we may assume that $L = D + N$, where D is diagonal, N is nilpotent and $DN = ND$. By taking $f^2 = f \circ f$ if necessary, we may assume that $\mu > 0$.

Definition 2.3. A holomorphic polynomial map $G\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ is said to satisfy the *resonance condition* with respect to f, if $G \circ D = D \circ G$.

The following lemma is given in [6].

Lemma 2.4. *Let $f : M \to M$ be a germ of a C^k, $k \geq |\tau|+1$, CR diffeomorphism such that $f(0) = 0$. Then there exists a local biholomorphic map ϕ of $\mathbb{C}^{n+1}$ such that*

(i) $\phi(0) = 0$,
(ii) $d\phi(0) = id$,
(iii) $S_{\varepsilon^{-1}} \circ \phi \circ S_\varepsilon = O(1)$ *as* $\varepsilon \to 0$,
(iv) $\phi \circ f \circ \phi^{-1} = (\mu w, Lz) + (0, R(z)) + o(|z|^k)$, *where* $L = D + N$ *with* $DN = ND$ *and* $R : \mathbb{C}^n \to \mathbb{C}^n$ *is a polynomial map satisfying the resonance condition with respect to* f.

Since ϕ satisfies $S_{\varepsilon^{-1}} \circ \phi \circ S_\varepsilon = O(1)$ as $\varepsilon \to 0$, each component R^j of $R(z)$ is of weight greater or equal to m_j and the image $\phi(M)$ is also defined by

$$\rho = \text{Im } w - P(z, \overline{z}) + r(z, \overline{z}, \text{Re } w),$$

where $P(z, \overline{z})$ is a weighted homogeneous polynomial of weight 1, i.e.,

$$P(\lambda^{m_1} z_1, \ldots \lambda^{m_n} z_n, \lambda^{m_1} \bar{z}_1, \ldots \lambda^{m_n} \bar{z}_n) = \lambda P(z, \overline{z})$$

without any pluri-harmonic terms and $r(z, \bar{z}, \text{Re } w)$ is a C^∞ function whose formal power series consists of terms of of higher weights.

3. Infinitesimal CR automorphisms

Let Ω, $p \in \partial\Omega$ and f be as in Theorem 1.1. Assume that $|\tau| < \infty$. In this section we show the existence of infinitesimal CR automorphism of $\partial\Omega$ near p transversal to complex tangent space, i.e., we show that $\partial\Omega$ is rigid near p.

Definition 3.1. A germ of a C^1 real vector field T tangent to a real hypersurface $M \subset \mathbb{C}^{n+1}$ is called an *infinitesimal CR automorphism* of M if for any $(1,0)$-vector field X of M, $[T, X]$ is again a $(1,0)$-vector field.

It is known that T is an infinitesimal CR automorphism of M if and only if T is of the form

$$T = \text{Re} \sum_{j=0}^{n} g^j \frac{\partial}{\partial z_j},$$

where g^j is a CR function, i.e., g^j satisfies

$$\overline{X} g^j = 0$$

for all $(1,0)$-vector field X of M.([1])

Let

$$f = (\mu w, Dz + Nz) + o(|(w,z)|^1),$$

where D is diagonal and N is nilpotent such that $DN = ND$ as in §1. Write

$$D = diag(\lambda_1, \dots, \lambda_n).$$

Note that we assumed $\mu > 0$.

Proposition 3.2. *There exists a C^{k-1} infinitesimal CR automorphism T of $\partial\Omega$ such that $T(p) \perp T_p^{1,0}\partial\Omega$ and $f_*(T) = \mu T$. Furthermore, we have*

$$\mu < |\lambda_j|, \ \forall j = 1, \dots, n.$$

Proof. Since f is a CR map on $\partial\Omega$, we may extend df_p to $T_p^{1,0}\mathbb{C}^{n+1}$ as a complex linear map. Let λ be an eigenvalue of df_p with the smallest absolute value. Since f is a contraction at 0, we can choose positive real numbers r, s and an open neighborhood U of 0 such that

(i) $s < |\lambda|$, $|\lambda| r/s < 1$.
(ii) If $x \in U \cap \overline{\Omega}$, then $\|f(x)\| < r\|x\|$ and $\|df(x)\| < r$.
(iii) If $x \in f(U \cap \overline{\Omega})$, then $\|df_x^{-1}\| \le \dfrac{1}{s}$.

Choose a holomorphic vector field $X_0(x)$ in a neighborhood of p such that Re $X_0(p)$ is tangent to $\partial\Omega$ at p and $X_0(p)$ is an eigenvector of λ. Define X_ν for $\nu \ge 1$ inductively by

$$X_\nu(x) = \lambda \left(df^{-1}(x)\right) X_{\nu-1}(f(x))$$

for all $x \in U \cap \overline{\Omega}$. First, we will show that $\{X_\nu\}$ converges to a C^{k-1} vector field X_∞ such that $X_\infty(p) = X_0(p)$. Clearly X_∞ is holomorphic in Ω and satisfies

$$f_*(X_\infty) = \lambda X_\infty.$$

For $\nu \geq 1$, let

$$Y_\nu = X_\nu - X_{\nu-1}.$$

Since $Y_1(x) = \alpha(x)X_0(x)$ for some C^{k-1} function $\alpha(x)$ vanishing at 0, we can show that

$$\|Y_{\nu+1}(x)\| = \|\lambda^\nu (df)^{-\nu} \phi(f^\nu(x))\| \leq C_0 \left(\frac{|\lambda| r}{s}\right)^\nu$$

for some constant C_0. Since $\dfrac{|\lambda| r}{s} < 1$, we can show that X_ν converges uniformly in $U \cap \overline{\Omega}$.

We claim that for any nonnegative integer $l \leq k-1$, there exist positive constants C_l and r_l with $r_l < 1$ such that

$$\left\| Y_{\nu+1}^{(l)}(x) \right\| \leq C_l \, r_l^\nu$$

for all ν, where $Y_\nu^{(l)}$ is the l-th order derivatives of Y_ν. We have seen that the claim holds for $l = 0$ with $r_0 = \dfrac{|\lambda| r}{s}$. Assume that the claim holds for $l < l_0$. Since

$$Y_{\nu+1} = \lambda df^{-1}(x) Y_\nu(f(x)),$$

we have

$$Y_{\nu+1}^{(l_0)}(x) = \lambda \left(df^{-1}(x)\right) Y_\nu^{(l_0)}(f(x))(df(x))^{l_0} + \lambda H\left(Y_\nu^{(l_0-1)}(f(x)), f^{(l_0+1)}\right),$$

where H is a polynomial independent of ν. Since f is C^k, we can choose a constant C independently of ν such that

$$\left\| H\left(Y_\nu^{(l_0-1)}(f(x)), f^{(l_0+1)}\right) \right\| \leq C \left\| Y_\nu^{(l_0-1)}(f(x)) \right\|.$$

By induction argument, there exist C_{l_0-1} and r_{l_0-1} such that

$$\left\| Y_\nu^{(l_0-1)}(f(x)) \right\| \leq C_{l_0-1} \, r_{l_0-1}^{\nu-1}.$$

Therefore for another constant $\tilde{C}$, we have

$$\left\| H\left(Y_\nu^{(l_0-1)}(f(x)), f^{(l_0+1)}\right) \right\| \leq \tilde{C} \, r_{l_0-1}^{\nu-1}.$$

Since

$$\left\| \lambda \left(df(x)^{-1}\right) Y_\nu^{(l_0)}(f(x))(df(x))^{l_0} \right\| \leq \left(\frac{|\lambda| r}{s}\right) \left\| Y_\nu^{(l_0)} \right\|$$

and since $\dfrac{|\lambda|r}{s} < 1$, we can choose C_l and $r_l < 1$ with $r_l > \max(r_{l_0-1}, \dfrac{|\lambda|r}{s})$ such that

$$\left\| Y^{(l)}_{\nu+1}(x) \right\| \le C_l\ r_l^{\nu}.$$

Since $r_l < 1$, from the claim above, we can deduce that $Y^{(l)}_{\nu}$ converges uniformly to 0 as $\nu \to \infty$ for all $l \le k-1$. Therefore we conclude that X_∞ is C^{k-1}. Now let

$$T := \text{Re } X_\infty.$$

Since T is the real part of $(1,0)$-vector field which is holomorphic in Ω, if T is tangent to $\partial\Omega$, then T is an infinitesimal CR automorphism of $\partial\Omega$.

We will show that T is tangent to $\partial\Omega$. Suppose that $X_\infty(p) \perp T^{1,0}_p\partial\Omega$, i.e., $T(p) \perp T^{1,0}_p\partial\Omega$. In this case, as in Theorem 5 of [6], we can show that T is tangent to $\partial\Omega$. Suppose that $X_\infty \in T^{1,0}_p\partial\Omega$. Choose a $(1,0)$ vector field Y transversal to $T^{1,0}\partial\Omega$ and choose a complex valued continuous function θ vanishing at p such that $X_\infty + \theta N$ is tangent to $\partial\Omega$. Then we have

$$f_*(X_\infty(x) + \theta(x)Y(x)) = f_*(X_\infty(x)) + \theta(x) f_*(Y(x)).$$

On the other hand, by the property of X_∞, we have

$$f_*(X_\infty(x)) = \lambda X_\infty(f(x)).$$

Since Y is normal to $\partial\Omega$ and $X_\infty + \theta Y$ is tangent to $\partial\Omega$, by comparing the coefficients, we can show that $|\theta(x)| = |\lambda\theta(f(x))|$. Then we have

$$|\theta(x)| = \lim_{\nu\to\infty} |\lambda^\nu \theta(f^\nu(x)|.$$

Since θ is continuous and $f^\nu(x) \to 0$ as $\nu \to \infty$, this implies that

$$\theta(x) \equiv \theta(0) = 0.$$

Hence we have that X_∞ is tangent to $\partial\Omega$, i.e., Re X_∞ and Im X_∞ both are tangent to $\partial\Omega$..

Finally we will show that $\mu < |\lambda_j|$ for $j = 1, \ldots, n$. Suppose not. Then we can construct a nowhere vanishing $(1,0)$-vector field X_∞ near p which is tangent to $\partial\Omega$ such that

$$[X_\infty, \overline{X}] \equiv 0$$

modulo $T^{0,1}\partial\Omega$ for all $(1,0)$ vector field X. Therefore we can choose a holomorphic vector field $\widetilde{X}$ with $\widetilde{X}(0) \ne 0$ such that

$$X_\infty = \widetilde{X} + o(|x|^{k-1}).$$

Then we have

$$\widetilde{X}\rho = o(|x|^{k-1}),$$

where ρ is a defining function for $\partial\Omega$ at p. Hence we have $\tau_n > k-1$, which is a contradiction. □

It is shown by Tanaka in [11] that a germ of a nowhere-vanishing C^∞ infinitesimal CR automorphism transversal to CR structure bundle can be straightened by a CR transform. Similar to C^∞ case, we can straighten C^{k-1} infinitesimal CR automorphism. More precisely, we can prove the following proposition. The proof is a modification of [2]. See Lemma I.1 of [2] for reference.

LEMMA 3.3. *There exists a C^k CR diffeomorphism $\psi : (\partial\Omega, p) \to (\psi(\partial\Omega), 0)$ such that*

$$\psi_* T = \mathrm{Re}\,\frac{\partial}{\partial w}.$$

Proof. Let $M = \partial\Omega$. Since T is an infinitesimal CR vector filed, there exists a 1-parameter family of C^{k-1} CR diffeomorphisms $\Phi : (-\epsilon, \epsilon) \times M \to M$ such that

$$\begin{cases} \dfrac{\partial}{\partial t}\Phi(t,x) = T(\Phi(t,x)) \\ \Phi(0,x) = x \end{cases},$$

where $t \in (-\epsilon, \epsilon)$ and $x \in M$. Let $e_1, \ldots, e_{2n}$ be the smooth real vector fields which generate the CR structure bundle $TM \cap JTM$, where J is the standard complex structure on $\mathbb{C}^{n+1}$. Consider the flow maps $X_1, \ldots, X_{2n}$ of $e_1, \ldots, e_{2n}$. For $y \in \mathbb{R}^{2n}$, sufficiently close to 0, define

$$X(y) = X_{2n}\left(y_{2n}, X_{2n-1}\left(y_{2n-1}, \cdots, X_2\left(y_2, X_1(y_1, 0)\right)\cdots\right)\right).$$

Choose local coordinates $(t, y) = (\mathrm{Re}\, w, y_1, \ldots, y_{2n})$ for M, where $t \in \mathbb{R}$ and $y \in \mathbb{R}^{2n}$ given by a coordinate chart $(t, y) \to \Phi(t, X(y))$. Since $T(0) \neq 0$, this map is a diffeomorphism for a sufficiently small neighborhood of $(0,0) \in \mathbb{R} \times \mathbb{R}^{2n}$. In this coordinates, we have $T = \dfrac{\partial}{\partial t}$. Furthermore, there exist coordinates

$$z_j = z_j(y_1, \ldots, y_{2n}),\ j = 1, \ldots, n,$$

C^{k-1} in y such that the vector fields

$$X_j = \frac{\partial}{\partial z_j} + \sum_{j=1}^{n} d_j^k(z, \bar{z}, t)\frac{\partial}{\partial \bar{z}_k} + h_j(z, \bar{z}, t)\frac{\partial}{\partial t},\ j = 1, \ldots, n$$

mutually commute with each other and form a CR basis of M, where $z = (z_1, \dots, z_n)$.

Since T is an infinitesimal CR automorphism, i.e., $[X_j, T] = 0$, we conclude that

$$h_j(z, \bar{z}, t) = h_j(z, \bar{z}, 0) =: a_j(z, \bar{z})$$

for all j and k. Put $w = t + \sqrt{-1}\ r$, where r is the solution to the equation

$$\frac{\partial}{\partial z_j} r(z, \bar{z}) = a_j(z, \bar{z}),\ j = 1, \dots, n$$

with $r(0) = 0$. Then the map ψ defined by

$$\psi(t, z) = (w, z_1, \dots, z_n)$$

is the desired CR diffeomorphism. □

PROPOSITION 3.4. *Let* $\frac{\partial}{\partial z_j}$ *be an eigenvector of* $L = df_p|_{T_p^{1,0}\partial\Omega}$ *with eigenvalue* λ_j. *Then there exist a sequence of points* $\{q_\nu\} \subset \Omega$ *with* $\|q_\nu - p\| \sim \mu^\nu$ *and a sequence of complex curves* $\{\mathcal{C}_\nu\} \subset \Omega$ *such that*

(i) $\mathcal{C}_\nu$ *is the image* $\xi_\nu(\Delta_\nu)$ *of a holomorphic map*

$$\xi_\nu : \Delta_\nu := \{\zeta \in \mathbb{C} : |\zeta| < C_1 \|q_\nu - p\|^{m_j}\} \to \Omega$$

such that $\xi_\nu(0) = q_\nu$ *and* $\|d\xi_\nu(0)\| \geq C_2$, *where* C_1 *and* C_2 *are constants independent of* ν.

(ii) $f(q_\nu) = q_{\nu+1}$.

(iii) $d(f \circ \xi_\nu)\left(\frac{\partial}{\partial \zeta}\Big|_0\right) = \lambda_j\ d\xi_{\nu+1}\left(\frac{\partial}{\partial \zeta}\Big|_0\right)$.

Proof. Let ψ be a germ of a biholomorphic map as in Lemma 3.3. Then $\psi_*(T) = \mathrm{Re}\ \frac{\partial}{\partial w}$ is tangent to $\partial\Omega$, which implies that

$$\psi(\partial\Omega) = \{(w, z) \in \mathbb{C} \times \mathbb{C}^n : \mathrm{Im}\ w = r(z, \bar{z})\},$$

for some real valued C^k function r. After a holomorphic change of coordinates, we may assume that $\psi(\partial\Omega)$ is defined by

$$\mathrm{Im}\ w = P(z, \bar{z}) + \widetilde{r}(z, \bar{z}),$$

where P is a weighted homogeneous polynomial of weight 1 as in §1 and $\widetilde{r}$ is a C^k function of higher weights. Since r is C^k and $\widetilde{r}$ is of weight strictly larger than 1, there exists a constant C such that

$$P(0, \cdots, 0, z_j, 0, \cdots, 0, 0, \cdots, 0, \bar{z}_j, 0, \cdots, 0) \leq \frac{1}{2} C\ |z_j|^{\tau_j} \tag{3.1}$$

and

$$\widetilde{r}(0,\cdots,0,z_j,0,\cdots,0,0,\cdots,0,\bar{z}_j,0,\cdots,0) \leq \frac{1}{2}C\,|z_j|^{\tau_j+1}. \tag{3.2}$$

Since f is a holomorphic map in Ω, C^k up to $\partial\Omega$ such that $f_*(T) = \mu T$, we have

$$\widetilde{f} := \psi^{-1}\circ f\circ\psi = (\mu w, H(z))$$

for some holomorphic map $H : \mathbb{C}^n \to \mathbb{C}^n$. Assume that

$$q := (-\sqrt{-1}, 0, \cdots, 0) \in \psi(\Omega).$$

Then a sequence defined by

$$q_\nu := (-\mu^\nu\sqrt{-1}, 0, \dots, 0),\ \nu \geq 0,$$

satisfies $\widetilde{f}(q_\nu) = q_{\nu+1}$. Let

$$\Delta_\nu := \{(0,\dots,0,z_j,0,\dots,0) + q_\nu : |z_j| < \widetilde{C}\ |q_\nu|^{m_j}\}$$

for some positive constant $\widetilde{C} \leq min(C^{-m_j}, j = 1,\dots,n)$. Then by (3.1) and (3.2), we have $\Delta_\nu \subset \psi(\Omega)$.

Now let $\dfrac{\partial}{\partial z_j}$ is an eigenvector of L with eigenvalue λ_j. After a linear change of coordinates in z, we may assume that it is also an eigenvector of dH_0 with eigenvalue λ_j. Hence we have

$$H(0,\cdots,0,z_j,0,\cdots,0) = (0,\cdots,0,\lambda_j z_j,0,\cdots,0) + o(|z_j|),$$

which implies

$$d\widetilde{f}|_{\Delta_\nu}(q_\nu) = (0,\cdots,0,\lambda_j,0,\cdots,0).$$

Hence the images $\psi(q_\nu)$ and $\psi(\Delta_\nu)$ are the desired sequences. □

4. Main technical theorem

Let Ω, $p \in \partial\Omega$ and f be as in Theorem 1.1. Assume that $p = 0$ and $|\tau| < \infty$. In this section we show the following theorem.

THEOREM 4.1. *Let $\partial\Omega$ is defined by*

$$\rho = \text{Im } w - P(z,\overline{z}) + r(z,\overline{z},\text{Re } w)$$

and let

$$f = (\mu w, Dz + Nz + R(z)) + o(|z|^k)$$

as in Lemma 2.4. Then it holds that

$$|\lambda_j| = \mu^{m_j},$$

where $D = diag(\lambda_1,\dots,\lambda_n)$.

Note that R both satisfies

$$S_\varepsilon^{-1} \circ (0, R) \circ S_\varepsilon \in O(1) \quad \text{as} \quad \varepsilon \to 0$$

and

$$D \circ R = R \circ D.$$

Proof of Theorem 4.1 :

Consider the equation

$$\rho \circ f(x) = 0, \ \forall x \in \partial\Omega$$

and comnpare the weights. Then we obtain

$$\mu P(z, \bar{z}) = P(Lz + R(z), \bar{L}\bar{z} + \bar{R}(\bar{z})). \tag{4.1}$$

We will prove Theorem 4.1 by induction. We will show that for $t \geq 1$, it holds that

$$|\lambda_j| = \mu^{m_j} \text{ for every } j = \ell_t, \ldots, \ell_{t+1} - 1.$$

Recall that

$$Z_t = (z_{\ell_t}, \ldots, z_{\ell_{t+1}-1}).$$

We let $(W_1, \ldots, W_s)$ be the complexification of $(\bar{Z}_1, \ldots, \bar{Z}_s)$.

4.1. Proof of Step (1)

Let $\widetilde{\lambda} := \min(|\lambda_i|, i = 1, \ldots, n)$.

Lemma 4.2. $\widetilde{\lambda} \leq \mu^{m_1}$.

Proof. Comparing in (4.1), terms with the smallest degree yield the identity

$$\mu P_1(Z_1, W_1) = P_1(L_1 Z_1, \bar{L}_1 W_1). \tag{4.2}$$

Let

$$P_1(Z_1, W_1) = \sum_{\alpha,\beta} a_{\alpha,\beta} (Z_1)^\alpha (W_1)^\beta.$$

Comparing the degrees by the lexicographic ordering, there exist multi-indices α and β with $|\alpha| + |\beta| = \tau_1 = m_1^{-1}$ such that

$$\mu = \lambda_1^{\alpha_1} \cdots \lambda_{\ell_2-1}^{\alpha_{\ell_2-1}} \bar{\lambda}_1^{\beta_1} \cdots \bar{\lambda}_{\ell_2-1}^{\beta_{\ell_2-1}}$$

Since $\widetilde{\lambda} = \min(|\lambda_1|, \ldots, |\lambda_n|)$, we have $\widetilde{\lambda} \leq \mu^{m_1}$. □

Lemma 4.3. $\widetilde{\lambda} = \mu^{m_1}$.

Proof. Suppose $\widetilde{\lambda} < \mu^{m_1}$. Assume that $\dfrac{\partial}{\partial z_j}$ is an eigenvector of L_1. Choose a sequence of points $q_\nu \in \Omega$ converging to 0 such that $|q_\nu| \sim \mu^\nu$ and a sequence of complex curves $\mathcal{C}_\nu = \xi_\nu(\Delta_\nu) \subset \Omega$ centered at q_ν as in Proposition 3.4. Let ρ_ν be the radius of Δ_ν. Consider the complex curves $f^{-\nu}(\mathcal{C}_\nu)$. Then

$$f^{-\nu}(\mathcal{C}_\nu) = \{f^{-\nu} \circ \xi_\nu(\rho_\nu \zeta) : \zeta \in \mathbb{C},\ |\zeta| < 1\}.$$

Since

$$df^{-\nu} \circ \xi_\nu(0) \geq C\ \widetilde{\lambda}^{-\nu}$$

and since $\rho_\nu \geq C|q_\nu|^{m_1}$ for some constant C, by passing to a subsequence of $f^{-\nu}(\mathcal{C}_\nu)$, we can construct a nonconstant complex curve $\xi : \mathbb{C} \to \Omega$, which contradicts the assumption that Ω is Kobayashi hyperbolic. □

LEMMA 4.4. $\widetilde{\lambda} = |\lambda_1| = \cdots = |\lambda_{\ell_2 - 1}|$.

Proof. Assume that

$$\widetilde{\lambda} = |\lambda_1| \leq \cdots \leq |\lambda_{\ell_2-1}|.$$

Suppose there exists j such that $\widetilde{\lambda} < |\lambda_j|$. Choose the smallest $j \leq \ell_2 - 1$ such that $\widetilde{\lambda} < |\lambda_j|$. We may assume that $\dfrac{\partial}{\partial z_j}$ is an eigenvector of L_1. By considering (4.2), we can show that there exists a multi-indices α and β with $|\alpha| + |\beta| = \tau_1 = m_1^{-1}$ and $\alpha_j + \beta_j \geq 1$ such that

$$\mu = \lambda_1^{\alpha_1} \cdots \lambda_{\ell_2-1}^{\alpha_{\ell_2-1}} \bar{\lambda}_1^{\beta_1} \cdots \bar{\lambda}_{\ell_2-1}^{\beta_{\ell_2-1}}.$$

But in Lemma 4.3, we showed that

$$|\lambda_l| \geq \mu^{m_1},\ l = 1, \ldots, \ell_2 - 1,$$

which yields a contradiction. □

4.2. Proof of Step (t+1) assuming Step (t)

Let $\widetilde{\lambda} := \min(|\lambda_i|, i = \ell_{t+1}, \ldots, n)$.

LEMMA 4.5. $\tilde{\lambda} \leq \lambda^{m_{\ell_{t+1}}}$.

Proof. By induction argument, we showed that

$$|\lambda_j| = \mu^{m_j}.$$

Since R satisfies the resonance condition with respect to f, i.e., $D \circ R = R \circ D$ for $Dz = (\lambda_1 z_1, \ldots, \lambda_n z_n)$, for each component R^j of R, we obtain

$$\begin{aligned} & R^j(\lambda^{m_1} z_1, \cdots, \lambda^{m_{\ell_{t+1}}} z_{\ell_{t+1}}, \cdots, \lambda^{m_{\ell_{t+2}-1}} z_{\ell_{t+2}-1}, 0) \\ &= R^j(\lambda^{m_1} z_1, \cdots, \lambda^{m_{\ell_{t+1}-1}} z_{\ell_{t+1}-1}, \frac{\lambda^{m_{\ell_{t+1}}}}{|\lambda_{\ell_{t+1}}|} |\lambda_{\ell_{t+1}}| z_{\ell_{t+1}}, \\ &\qquad \cdots, \frac{\lambda^{m_{\ell_{t+2}-1}}}{|\lambda_{\ell_{t+2}-1}|} |\lambda_{\ell_{t+2}-1}| z_{\ell_{t+2}-1}, 0) \\ &= \lambda^{m_j} R^j \left(z_1, \cdots, z_{\ell_t}, \frac{\lambda^{m_{\ell_{t+1}}}}{|\lambda_{\ell_{t+1}}|} z_{\ell_{t+1}}, \cdots, \frac{\lambda^{m_{\ell_{t+2}-1}}}{|\lambda_{\ell_{t+2}-1}|} z_{\ell_{t+2}-1}, 0 \right) \end{aligned}$$

for every $j \leq \ell_{t+1} - 1$.

Suppose $\tilde{\lambda} > \lambda^{m_{\ell_{t+1}}}$. Then $\dfrac{\lambda^{m_{\ell_{t+1}}}}{|\lambda_i|} < 1$ for every $i = \ell_{t+1}, \ldots, \ell_{t+2} - 1$. Since R^j is of weight m_j, this implies that R^j is independent of $z_{\ell_{t+1}}, \ldots, z_{\ell_{t+2}-1}$. Consider the equation (4.1). Then we get

$$\begin{aligned} & \mu P_{t+1}(Z_1, \ldots, Z_{t+1}, W_1, \ldots, W_{t+1}) \\ &\qquad = P_{t+1}(L_1 Z_1, \ldots, L_{t+1} Z_{t+1}, \bar{L}_1 W_1, \ldots, \bar{L}_{t+1} W_{t+1}). \end{aligned}$$

Replace P_{t+1} by its power series expansion and then compare the lexicographic ordering for Z_{t+1} and then for $Z_1, \ldots, Z_t$. Then there exist multi-indices α and β such that $|A_{t+1}| + |B_{t+1}| \neq 0$ and

$$\mu = \lambda^{m_1(\alpha_1+\beta_1)+\cdots+m_{\ell_{t+1}-1}(\alpha_{\ell_{t+1}-1}+\beta_{\ell_{t+1}-1})} \lambda_{\ell_{t+1}}^{\alpha_{\ell_{t+1}}} \cdots \bar{\lambda}_{\ell_{t+2}-1}^{\beta_{\ell_{t+2}-1}}.$$

Since $\lambda^{m_{\ell_{t+1}}} < \tilde{\lambda} \leq |\lambda_i|$ for all $i \geq \ell_{t+1}$, we have

$$\mu > \lambda^{m_1(\alpha_1+\beta_1)+\cdots+m_{\ell_{t+1}-1}(\alpha_{\ell_{t+1}-1}+\beta_{\ell_{t+1}-1})} |\lambda_{\ell_{t+1}}^{\alpha_{\ell_{t+1}}}| \cdots |\lambda_{\ell_{t+2}-1}^{\beta_{\ell_{t+2}-1}}| = \mu.$$

This is a contradiction. Hence we must have $\tilde{\lambda} \leq \lambda^{m_{\ell_{t+1}}}$. □

Similar to Lemma 4.3 and Lemma 4.4, we can prove the following.

Lemma 4.6. $\widetilde{\lambda} = \lambda^{m_{\ell_{t+1}}}$.

Lemma 4.7. $\widetilde{\lambda} = |\lambda_{\ell_{t+1}}| = \cdots = |\lambda_{\ell_{t+2}-1}|$.

5. Proof of Theorem 1.1

Let Ω, $p \in \partial\Omega$ and f be as in Theorem 1.1. In this section, we prove Theorem 1.1.

CASE 1. $|\tau| < \infty$.

Let T be an infinitesimal CR automorphism as in Lemma 3.2. Then by Lemma 3.3, there exist an open neighborhood U of p and a C^k map $\phi : \overline{\Omega} \cap U \to V \subset \mathbb{C}^{n+1}$, holomorphic in $\Omega \cap U$ such that $\phi_*(T) = \text{Re}\,\dfrac{\partial}{\partial w}$. Then we can show that

$$\phi(\partial\Omega \cap U) = \{(w, z) \in \mathbb{C} \times \mathbb{C}^n : \text{Im}\ w = P(z, \overline{z}) + \widetilde{r}(z, \bar{z})\} \cap V,$$

where P is a weighted homogeneous polynomial with weight 1, i.e.,

$$P(\varepsilon^{m_1} z_1, \ldots, \varepsilon^{m_n} z_n, \varepsilon^{m_1} \bar{z}_1, \ldots, \varepsilon^{m_n} \bar{z}_n) = \varepsilon P(z, \bar{z})$$

and $\widetilde{r}$ is a C^k real valued function with higher weights whose k-th order Taylor polynomial at 0 has no pluri-harmonic terms.

Let

$$\widetilde{f} := \phi^{-1} \circ f \circ \phi.$$

Since

$$\widetilde{f}_* \left(\text{Re}\ \frac{\partial}{\partial w} \right) = \mu\ \text{Re}\ \frac{\partial}{\partial w},$$

we obtain

$$\widetilde{f} = (\mu w + g(z), H(z))$$

for some $g(z)$ and $H(z)$ which are holomorphic. We may assume that

$$H(z) = (\lambda_1 z_1, \ldots, \lambda_n z_n) + Nz + R(z) + o(|z|^k)$$

for some nilpotent matrix N and a polynomial map R satisfying the resonance condition with respect to $\widetilde{f}$. Then by Theorem 4.1, we have such that

$$|\lambda_j| = \mu^{m_j}.$$

Since $\widetilde{f}$ is a contraction at 0, so is H. Since $|\tau| < k$, no nontrivial terms in $o(|z|^k)$ can satisfy the resonance condition with respect to $\widetilde{f}$. Therefore by Poincaré-Dulac Theorem([3]), we may assume that

$$H(z) = (\lambda_1 z_1, \ldots, \lambda_n z_n) + Nz + R(z).$$

By comparing the weights of the equation

$$\text{Im}\ (\mu w) = P(H(z), \overline{H(z)}) + \widetilde{r}(H(z), \overline{H(z)}),$$

we can show that

$$g(z),\ \widetilde{r}(z, \bar{z}) \in o(|z|^k).$$

Moreover, $\widetilde{r}$ satisfies

$$\mu\ \widetilde{r}(z, \bar{z}) + \text{Im}\ g(z) = \widetilde{r}(H(z), \overline{H(z)}).$$

Then we obtain

$$\widetilde{r}(z,\bar{z}) = \lim_{\nu\to\infty}\sum_{\ell=0}^{\nu-1} -\mu^{-\ell-1}\mathrm{Im}\; g(H^{\ell}(z)) + \mu^{-\nu}\;\widetilde{r}(H^{\nu}(z),\overline{H^{\nu}(z)}),$$

Choose a positive constant λ such that

$$\mu^{\tau_n^{-1}} < \lambda < \mu^{(\tau_n+1)^{-1}}.$$

Since $\tau_1 \le \cdots \le \tau_n$ and $m_j = \tau_j^{-1}$, we can show that

$$\|H(z)\| < \lambda\|z\|$$

for all z sufficiently close to 0. Since $k > \tau_n+1$ and $g(z),\ \widetilde{r}(z,\bar{z}) \in o(|z|^k)$, we obtain that $\sum_{\ell=0}^{\nu-1}\mu^{-\ell-1}\mathrm{Im}\; g(H^{\ell}(z))$ converges to a holomorphic function $\widetilde{g}(z)$ and $\mu^{-\nu}\;\widetilde{r}(H^{\nu}(z),\overline{H^{\nu}(z)})$ converges to 0 as $\nu\to\infty$. Then we have

$$\widetilde{r}(z,\bar{z}) = -\mathrm{Im}\;(\widetilde{g}(z)).$$

After a holomorphic change of coordinates, we may assume that

$$\phi(\partial\Omega\cap U) = \{(w,z)\in\mathbb{C}\times\mathbb{C}^n : \mathrm{Im}\; w = P(z,\overline{z})\}\cap V.$$

Let

$$\widetilde{\Omega} := \{(w,z)\in\mathbb{C}\times\mathbb{C}^n : \mathrm{Im}\; w < P(z,\overline{z})\}.$$

By Kobayashi hyperbolicity of Ω, we can show that the sequence of holomorphic map

$$\psi_\nu := \widetilde{f}^{-\nu}\circ\phi\circ f^{\nu}$$

has a subsequence which converges on compact subsets of Ω to a biholomorphic map $\phi : \Omega \to \widetilde{\Omega}$.

CASE 2. $|\tau| = \infty$.

By the assumption on f, we have $f \in C^\infty(\overline{\Omega})$. Then by the result of [7], we can show that there exist an open neighborhood U of p and a C^∞ map $\phi : \overline{\Omega}\cap U \to V \subset \mathbb{C}^{n+1}$ holomorphic in $\Omega\cap U$ with $\phi(p) = 0$ such that

$$\phi(\Omega\cap U) = \{(w,z)\in\mathbb{C}\times\mathbb{C}^n : \mathrm{Im}\; w < P(z,\overline{z})\}\cap V$$

for some weighted homogeneous polynomial P and $\widetilde{f} := \phi^{-1}\circ f\circ\phi$ satisfies

$$\widetilde{f}\circ\left(d\widetilde{f}_0\right) = \left(d\widetilde{f}_0\right)\circ\widetilde{f}.$$

Let

$$\widetilde{\Omega} := \{(w,z)\in\mathbb{C}\times\mathbb{C}^n : \mathrm{Im}\; w < P(z,\overline{z})\}.$$

By considering the sequence of holomorphic maps

$$\psi_\nu := \widetilde{f}^{-\nu}\circ\phi\circ f^{\nu}$$

as in case 1, we can show that there exists a biholomorphic map $\psi : \Omega \to \widetilde{\Omega}$, which completes the proof.

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