

## IRROTATIONAL SCREEN HOMOTHETIC HALF LIGHTLIKE SUBMANIFOLDS

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ABSTRACT. In this paper, we study the geometry of half lightlike submanifolds of a Lorentzian manifold. The main result is a characterization theorem for irrotational screen homothetic half lightlike submanifolds of a Lorentzian space form.

### 1. Introduction

It is well known that the radical distribution  $Rad(TM) = TM \cap TM^\perp$  of lightlike submanifolds  $M$  of a Lorentzian manifold  $(\bar{M}, \bar{g})$  is a vector subbundle of the tangent bundle  $TM$  and the normal bundle  $TM^\perp$ . A codimension 2 lightlike submanifold  $M$  of rank  $Rad(TM) = 1$  is called a half lightlike submanifold of  $(\bar{M}, \bar{g})$  [3, 5, 6, 7, 8]. Then there exists a complementary non-degenerate distribution  $S(TM)$  of  $Rad(TM)$  in  $TM$ , called a *screen distribution* on  $M$ , such that

$$(1.1) \quad TM = Rad(TM) \oplus_{orth} S(TM),$$

where the symbol  $\oplus_{orth}$  denotes the orthogonal direct sum. We denote such a half lightlike submanifold by  $(M, g, S(TM))$ . Denote by  $F(M)$  the algebra of smooth functions on  $M$  and by  $\Gamma(E)$  the  $F(M)$  module of smooth sections of a vector bundle  $E$  over  $M$ . Then there exist vector fields  $\xi \in \Gamma(Rad(TM))$  and  $u \in \Gamma(S(TM^\perp))$  such that

$$\bar{g}(u, u) = 1, \quad \bar{g}(\xi, v) = 0, \quad \forall v \in \Gamma(TM^\perp),$$

where  $S(TM^\perp)$  is a supplementary distribution to  $Rad(TM)$  in  $TM^\perp$  of rank 1, called a *co-screen distribution* on  $M$ . Consider the orthogonal complementary distribution  $S(TM)^\perp$  to  $S(TM)$  in  $T\bar{M}$ . Certainly  $\xi$  and  $u$  belong to  $\Gamma(S(TM)^\perp)$ . Thus we have

$$S(TM)^\perp = S(TM^\perp) \oplus_{orth} S(TM^\perp)^\perp,$$

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where  $S(TM^\perp)^\perp$  is a orthogonal complementary to  $S(TM^\perp)$  in  $S(TM)^\perp$ . For any section  $\xi$  of  $Rad(TM)$  on a coordinate neighborhood  $\mathcal{U} \subset M$ , there exists a unique vector field  $N \in \Gamma(ltr(TM))$  satisfying

$$(1.2) \quad \bar{g}(\xi, N) = 1, \quad \bar{g}(N, N) = \bar{g}(N, X) = \bar{g}(N, u) = 0,$$

for all  $X \in \Gamma(S(TM))$ . We call  $ltr(TM)$ ,  $tr(TM) = S(TM^\perp) \oplus_{orth} ltr(TM)$  and  $N$  the *lightlike transversal vector bundle*, *transversal vector bundle* and *lightlike transversal vector field* of  $M$  with respect to  $S(TM)$  respectively. Therefore  $T\bar{M}$  is decomposed as follows:

$$(1.3) \quad T\bar{M} = TM \oplus tr(TM) = \{Rad(TM) \oplus tr(TM)\} \oplus_{orth} S(TM) \\ = \{Rad(TM) \oplus ltr(TM)\} \oplus_{orth} S(TM^\perp) \oplus_{orth} S(TM).$$

The purpose of this paper is to study the geometry of half lightlike submanifolds of a Lorentzian manifold. We prove a characterization theorem for half lightlike submanifolds  $M$  of a Lorentzian space form  $(\bar{M}(c), \bar{g})$ ,  $c > 0$ : If  $M$  is irrotational and screen homothetic, then (1) the induced connection  $\nabla$  on  $M$  is a metric one and (2)  $M$  is totally umbilical and locally a product manifold  $M = L \times M^*$ , where  $L$  is a lightlike curve and  $M^*$  is a totally geodesic Riemannian space form which is isometric to a sphere (Theorem 2.3).

Let  $\bar{\nabla}$  be the Levi-Civita connection of  $\bar{M}$  and  $P$  the projection morphism of  $\Gamma(TM)$  on  $\Gamma(S(TM))$  with respect to the decomposition (1.1). Then the local Gauss and Weingarten formulas are given by

$$(1.4) \quad \bar{\nabla}_X Y = \nabla_X Y + B(X, Y)N + D(X, Y)u,$$

$$(1.5) \quad \bar{\nabla}_X N = -A_N X + \tau(X)N + \rho(X)u,$$

$$(1.6) \quad \bar{\nabla}_X u = -A_u X + \phi(X)N,$$

$$(1.7) \quad \nabla_X PY = \nabla_X^* PY + C(X, PY)\xi,$$

$$(1.8) \quad \nabla_X \xi = -A_\xi^* X - \tau(X)\xi,$$

for any  $X, Y \in \Gamma(TM)$ , where  $\nabla$  and  $\nabla^*$  are induced linear connections on  $TM$  and  $S(TM)$  respectively,  $B$  and  $D$  are called the local fundamental forms of  $M$ ,  $C$  is called the local second fundamental form on  $S(TM)$ .  $A_N$ ,  $A_\xi^*$  and  $A_u$  are linear operators on  $\Gamma(TM)$  and  $\tau$ ,  $\rho$  and  $\phi$  are 1-forms on  $TM$ . Since  $\bar{\nabla}$  is torsion-free,  $\nabla$  is also torsion-free and both  $B$  and  $D$  are symmetric. From the facts that  $B(X, Y) = \bar{g}(\bar{\nabla}_X Y, \xi)$  and  $D(X, Y) = \bar{g}(\bar{\nabla}_X Y, u)$ , we know that  $B$  and  $D$  are independent of the choice of a screen distribution and

$$(1.9) \quad B(X, \xi) = 0, \quad D(X, \xi) = -\phi(X), \quad \forall X \in \Gamma(TM).$$

The induced connection  $\nabla$  on  $M$  is not metric and satisfies

$$(1.10) \quad (\nabla_X g)(Y, Z) = B(X, Y)\eta(Z) + B(X, Z)\eta(Y),$$

for all  $X, Y, Z \in \Gamma(TM)$ , where  $\eta$  is a 1-form on  $TM$  such that

$$(1.11) \quad \eta(X) = \bar{g}(X, N), \quad \forall X \in \Gamma(TM).$$

But the connection  $\nabla^*$  on  $S(TM)$  is metric. The above three local second fundamental forms on  $TM$  and  $S(TM)$  are related to their shape operators by

$$(1.12) \quad B(X, Y) = g(A_\xi^* X, Y), \quad \bar{g}(A_\xi^* X, N) = 0,$$

$$(1.13) \quad C(X, PY) = g(A_N X, PY), \quad \bar{g}(A_N X, N) = 0,$$

$$(1.14) \quad D(X, PY) = g(A_u X, PY), \quad \bar{g}(A_u X, N) = \rho(X),$$

$$(1.15) \quad D(X, Y) = g(A_u X, Y) - \phi(X)\eta(Y).$$

From (1.12),  $A_\xi^*$  is a self-adjoint operator on  $\Gamma(TM)$  satisfying

$$(1.16) \quad A_\xi^* \xi = 0.$$

We denote by  $\bar{R}$ ,  $R$  and  $R^*$  the curvature tensors of  $\bar{\nabla}$ ,  $\nabla$  and  $\nabla^*$  respectively. Using the Gauss-Weingarten equations for  $M$  and  $S(TM)$ , we obtain the Gauss-Codazzi equations for  $M$  and  $S(TM)$ :

$$(1.17) \quad \begin{aligned} \bar{g}(\bar{R}(X, Y)Z, PW) &= g(R(X, Y)Z, PW) \\ &\quad + B(X, Z)C(Y, PW) - B(Y, Z)C(X, PW) \\ &\quad + D(X, Z)D(Y, PW) - D(Y, Z)D(X, PW), \end{aligned}$$

$$(1.18) \quad \begin{aligned} \bar{g}(\bar{R}(X, Y)Z, \xi) &= (\nabla_X B)(Y, Z) - (\nabla_Y B)(X, Z) \\ &\quad + B(Y, Z)\tau(X) - B(X, Z)\tau(Y) \\ &\quad + D(Y, Z)\phi(X) - D(X, Z)\phi(Y), \end{aligned}$$

$$(1.19) \quad \begin{aligned} \bar{g}(\bar{R}(X, Y)Z, N) &= \bar{g}(R(X, Y)Z, N) \\ &\quad + D(X, Z)\rho(Y) - D(Y, Z)\rho(X), \end{aligned}$$

$$(1.20) \quad \begin{aligned} \bar{g}(\bar{R}(X, Y)Z, u) &= (\nabla_X D)(Y, Z) - (\nabla_Y D)(X, Z) \\ &\quad + B(Y, Z)\rho(X) - B(X, Z)\rho(Y), \end{aligned}$$

$$(1.21) \quad \begin{aligned} \bar{g}(R(X, Y)PZ, PW) &= g(R^*(X, Y)PZ, PW) \\ &\quad + C(X, PZ)B(Y, PW) - C(Y, PZ)B(X, PW), \end{aligned}$$

$$(1.22) \quad \begin{aligned} g(R(X, Y)PZ, N) &= (\nabla_X C)(Y, PZ) - (\nabla_Y C)(X, PZ) \\ &\quad + C(X, PZ)\tau(Y) - C(Y, PZ)\tau(X). \end{aligned}$$

The induced Ricci type tensor  $R^{(0,2)}$  of  $M$  is given by

$$(1.23) \quad R^{(0,2)}(X, Y) = \text{trace}\{Z \rightarrow R(Z, X)Y\}, \quad \forall X, Y \in \Gamma(TM).$$

In general,  $R^{(0,2)}$  is not symmetric. A tensor field  $R^{(0,2)}$  is called its *induced Ricci tensor*, denoted by  $Ric$ , if it is symmetric.

## 2. Irrotational screen homothetic half lightlike submanifolds

DEFINITION 2.1. (a) A half lightlike submanifold  $(M, g, S(TM))$  of a Lorentzian manifold  $(\bar{M}, \bar{g})$  is called *screen homothetic* [1] if  $A_N = b A_\xi^*$  for a non-zero constant  $b$  on  $M$ , or equivalently,

$$(2.1) \quad C(X, PY) = b B(X, Y) \quad \forall X, Y \in \Gamma(TM).$$

(b)  $M$  is *irrotational* [9] if  $\bar{\nabla}_X \xi \in \Gamma(TM)$  for any  $X \in \Gamma(TM)$ .

NOTE 2.2. If  $M$  is screen homothetic, then  $C$  is symmetric on  $S(TM)$ . Therefore  $S(TM)$  is integrable [4]. Thus  $M$  is locally a product manifold  $L \times M^*$ , where  $L$  is a lightlike curve and  $M^*$  is a leaf of  $S(TM)$  [4]. If  $M$  is irrotational, then, from (1.4) and (1.9), we show that  $D(X, \xi) = 0$  and  $\phi(X) = 0$  for all  $X \in \Gamma(TM)$ .

THEOREM 2.3. Let  $(M, g, S(TM))$  be an irrotational screen homothetic half lightlike submanifold of  $(\bar{M}(c), \bar{g})$ . Then the induced Ricci type tensor  $R^{(0,2)}$  is an induced Ricci tensor of  $M$ .

*Proof.* Since  $(\bar{M}(c), \bar{g})$  is a Lorentzian space form, we obtain [5, 6]

$$\begin{aligned} R^{(0,2)}(X, Y) &= mcg(X, Y) + B(X, Y)tr A_N + D(X, Y)tr A_u \\ &\quad - g(A_N X, A_\xi^* Y) - g(A_u X, A_u Y) + \rho(X)\phi(Y). \end{aligned}$$

From this and the facts  $A_N = b A_\xi^*$  and  $\phi = 0$ , we have our assertion.  $\square$

THEOREM 2.4. [5] Let  $(M, g, S(TM))$  be a half lightlike submanifold of a Lorentzian manifold  $(\bar{M}, \bar{g})$ . If  $R^{(0,2)}$  is symmetric, then there exists a pair  $\{\xi, N\}$  on  $\mathcal{U}$  such that the 1-form  $\tau$  vanishes on  $M$ .

We call a pair  $\{\xi, N\}$  on  $\mathcal{U}$  such that the 1-form  $\tau$  vanishes on  $M$  the *distinguished null pair* of  $M$ . Although, in general,  $S(TM)$  is not unique, it is isomorphic to the factor bundle  $S(TM)^* = TM/Rad(TM)$  considered by Kupeli [9]. Thus all  $S(TM)$  are mutually isomorphic. For this reason, in the sequel, let  $(M, g, S(TM))$  be an irrotational screen homothetic half lightlike submanifold of a Lorentzian space form  $(\bar{M}(c), \bar{g})$  equipped with the distinguished null pair  $\{\xi, N\}$ .

THEOREM 2.5. Let  $(M, g, S(TM))$  be an irrotational screen homothetic half lightlike submanifold of a Lorentzian space form  $(\bar{M}(c), \bar{g})$ . If  $\dim M > 3$ , then we have the following two assertions:

- (1) *The induced connection  $\nabla$  on  $M$  is a metric connection.*
- (2)  *$M$  is totally umbilical and locally a product manifold  $L \times M^*$ , where  $L$  is a null curve and  $M^*$  is a totally geodesic Riemannian space form which is isometric to a sphere.*

*Proof.* Since  $(\bar{M}(c), \bar{g})$  is a space of constant curvature  $c$  and  $M$  is an irrotational screen homothetic, from (1.18), we obtain

$$(2.2) \quad (\nabla_X B)(Y, Z) = (\nabla_Y B)(X, Z)$$

for all  $X, Y, Z \in \Gamma(TM)$ . Using this, (1.19), (1.22) and (2.1), we get

$$(2.3) \quad \begin{aligned} \rho(Y)D(X, PZ) - \rho(X)D(Y, PZ) \\ = c \{g(Y, PZ)\eta(X) - g(X, PZ)\eta(Y)\}. \end{aligned}$$

Replace  $Y$  by  $\xi$  in (2.3) and use (1.9) with  $\phi = 0$ , we obtain

$$(2.4) \quad \rho(\xi)D(X, PZ) = -c g(X, PZ).$$

Since  $c \neq 0$ , we have  $\rho(\xi) \neq 0$  and  $D \neq 0$ . From (1.9) and (2.4), we get

$$(2.5) \quad D(X, Y) = \alpha g(X, Y), \quad \forall X, Y \in \Gamma(TM),$$

where  $\alpha = -c\rho(\xi)^{-1} \neq 0$ . While, from (1.20), we get

$$(2.6) \quad (\nabla_X D)(Y, Z) - (\nabla_Y D)(X, Z) + B(Y, Z)\rho(X) - B(X, Z)\rho(Y) = 0,$$

for all  $X, Y, Z \in \Gamma(TM)$ . From this, (1.10) and (2.5), we have

$$(2.7) \quad \begin{aligned} X[\alpha]g(Y, Z) - Y[\alpha]g(X, Z) + B(X, Z)\{\alpha\eta(Y) - \rho(Y)\} \\ - B(Y, Z)\{\alpha\eta(X) - \rho(X)\} = 0, \quad \forall X, Y, Z \in \Gamma(TM). \end{aligned}$$

Replace  $Y$  by  $\xi$  in (2.7) and use (1.9), we obtain

$$(2.8) \quad B(X, Y)\{\alpha - \rho(\xi)\} = \xi[\alpha]g(X, Y).$$

Since  $c > 0$ , we have  $\alpha - \rho(\xi) \neq 0$ . From (2.1) and (2.8), we have

$$(2.9) \quad B(X, Y) = \beta g(X, Y), \quad C(X, Y) = b\beta g(X, Y),$$

for all  $X, Y \in \Gamma(TM)$ , where  $\beta = \xi[\alpha](\alpha - \rho(\xi))^{-1}$ . Thus  $M$  and  $S(TM)$  are totally umbilical. Since  $M$  is screen homothetic, by Note 1,  $M$  is locally a product manifold  $L \times M^*$  where  $L$  is a lightlike curve and  $M^*$  is a leaf of  $S(TM)$ . Since  $\bar{M}$  is a space of constant curvature  $c$ , from (1.17), (1.21) and (2.9), we have

$$(2.10) \quad R^*(X, Y)Z = (c + \alpha^2 + 2b\beta^2)\{g(Y, Z)X - g(X, Z)Y\},$$

for all  $X, Y, Z \in \Gamma(S(TM))$ . Let  $Ric^*$  be the induced symmetric Ricci tensor of  $M^*$ . From (2.10), we have

$$Ric^*(X, Y) = (m - 1)(c + \alpha^2 + 2b\beta^2)g(X, Y), \quad \forall X, Y \in \Gamma(S(TM)).$$

Thus  $M^*$  is an Einstein manifold. Since  $M^*$  is a Riemannian manifold with  $\dim M^* > 2$ ,  $(c + \alpha^2 + 2b\beta^2)$  is a constant and  $M^*$  is a space of constant curvature. Differentiating the first equation of (2.9) and using (1.10), (2.2) and the first equation of (2.9), we have

$$\{X[\beta] - \beta^2\eta(X)\}g(Y, Z) = \{Y[\beta] - \beta^2\eta(Y)\}g(X, Z).$$

Replace  $Y$  by  $\xi$  in this equation, we have  $\xi[\beta] = \beta^2$ . Since  $(c + \alpha^2 + 2b\beta^2)$  is a constant,  $0 = \xi[c + \alpha^2 + 2b\beta^2] = 2\beta(c + \alpha^2 + 2b\beta^2)$ . Since  $(c + \alpha^2 + 2b\beta^2)$  is a constant, we have  $\beta = 0$  or  $c + \alpha^2 + 2b\beta^2 = 0$ . If  $c + \alpha^2 + 2b\beta^2 = 0$ ,  $M^*$  is a Euclidean space and the second fundamental form  $C$  of  $M^*$  satisfies  $C = 0$  [2]. From (2.9), we have  $b\beta = 0$ . Thus we have  $c + \alpha^2 = 0$ . This means  $c = 0$  and  $\alpha = 0$ . It is a contradiction to  $c > 0$  and  $\alpha \neq 0$ . Consequently, we have  $\beta = 0$  and  $c + \alpha^2 \neq 0$ . Thus  $B = C = 0$ ;  $D \neq 0$  and  $M^*$  is a totally geodesic Riemannian space form of positive constant curvature  $(c + \alpha^2)$  which is isometric to a sphere. Since  $B(X, Y) = 0$  for all  $X, Y \in \Gamma(TM)$ , we show that the induced connection  $\nabla$  is metric due to (1.10).  $\square$

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