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ON THE q–EXTENSION OF THE HARDY–LITTLEWOOD–TYPE MAXIMAL OPERATOR RELATED TO q–VOLKENBORN INTEGRAL IN THE p–ADIC INTEGER RING

Lee-Chae Jang*

Abstract. In this paper, we define the q-extension of the Hardy-Littlewood-type maximal operator related to q-Volkenborn integral. By the meaning of the extension of q-Volkenborn integral, we obtain the boundedness of the q-extension of the Hardy-Littlewood-type maximal operator in the p-adic integer ring.

1. Introduction and preliminaries

Let p be a fixed odd prime. Throughout this paper $\mathbb{Z}, \mathbb{Q}, \mathbb{Z}_p, \mathbb{Q}_p$ and $\mathbb{C}_p$ will, respectively, denote the ring of rational integers, the field of rational integers, the ring of p-adic rational integers, the field of p-adic rational numbers and the completion of algebraic closure of $\mathbb{Q}_p$. Let v_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-v_p(p)} = p^{-1}$. When one talks of q-extension, q is variously considered as an indeterminate, a complex number $q \in \mathbb{C}$, or a p-adic number $q \in \mathbb{C}_p$, cf. [1-5, 17-20]. In this paper, we assume that $q \in \mathbb{C}_p, |1-q|_p < 1$. We also use the notation

$$[x]_q = \frac{1-q^x}{1-q},$$

for all $x \in \mathbb{Z}_p$. Hence, $\lim_{q\to 1}[x]_q = x$.

Let d be a fixed positive integer with $(p, d) = 1$. We now set

$$X = \varprojlim_N \mathbb{Z}/dp^N\mathbb{Z},$$

$$X^* = \bigcup_{\substack{0<a<dp \\ (a,p)=1}} a + dp\mathbb{Z}_p,$$

$$a + dp^N\mathbb{Z}_p = \{x \in X | x \equiv a \pmod{p^N}\},$$

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where $a \in \mathbb{Z}$ lies in $0 \leq a < dp^N$. For any $N \in \mathbb{N}$, we set

$$\mu_q(a + dp^N \mathbb{Z}_p) = \frac{q^a}{[dp^N]_q}$$

and this can be extended to a distribution on $\mathbb{Z}_p$. We recall that μ_q is called p-adic q-invariant distribution on $\mathbb{Z}_p$.

Let $C(\mathbb{Z}_p, \mathbb{C}_p)$ be the space of continuous function on $\mathbb{Z}_p$ with values in $\mathbb{C}_p$ with supnorm $\| f \|_\infty = \sup_{x \in \mathbb{Z}_p} |f(x)|_p$. The difference quotient $\Delta_1 f$ of f is the function of two variables given by

$$\Delta_1 f(m, x) = \frac{f(x+m) - f(x)}{m},$$

for all $x, m \in \mathbb{Z}_p,\ m \neq 0$. A function $f : \mathbb{Z}_p \to \mathbb{C}_p$ is said to be a Lipschitz function if there exists a constant $M > 0$ (the Lipschitz constant of f) such that

$$|\Delta_1 f(m, x)| \leq M,$$

for all $m \in \mathbb{Z}_p \backslash \{0\}$ and $x \in \mathbb{Z}_p$. The $\mathbb{C}_p$-linear space consisting of all Lipschitz function (or $C^{(1)}$-function) is denoted by $\mathrm{Lip}(\mathbb{Z}_p, \mathbb{C}_p)$ (or $C^{(1)}(\mathbb{Z}_p, \mathbb{C}_p)$. This space is a Banach space with respect to the norm $\|f\|_1 = \|f\|_\infty \vee \|\Delta_1 f\|_\infty$ (see [13]).

For $f \in C^{(1)}(\mathbb{Z}_p, \mathbb{C}_p)$, the q-Volkenborn integral is defined by

$$I_q(f) = \int_{\mathbb{Z}_p} f(x) d\mu_q(x) = \lim_{N \to \infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N - 1} f(x) q^x \tag{1.1}$$

(see [1-15, 17-20]). By the meaning of the extension of q-Volkenborn integral, we consider the below weakly (strongly) p-adic q-invariant distribution μ_q on $\mathbb{Z}_p$ satisfying

$$|[p^n]_q \mu_q(a + p^n \mathbb{Z}_p) - [p^{n+1}]_q \mu_q(a + p^{n+1} \mathbb{Z}_p)|_p \leq \delta_n, \tag{1.2}$$

where $\delta_n \to 0, a \in \mathbb{Z}$, and δ_n is independent of a (for strong p-adic q-invariant distribution, δ_n is replaced by cp^{-n}, where c is positive real constant). Let $f \in C^{(1)}(\mathbb{Z}_p, \mathbb{C}_p)$. For any $a \in \mathbb{Z}_p$, define

$$\mu_{f,q}(a + p^n \mathbb{Z}_p) = \int_{a + p^n \mathbb{Z}_p} f(x) d\mu_q(x), \tag{1.3}$$

where the integral is the extension of q-Volkenborn integral.

The purpose of this paper is to define the q-extension of the Hardy-Littlewood-type maximal operator related to q-Volkenborn integral and to obtain the boundedness of the q-extension of the Hardy-Littlewood-type maximal operator in the p-adic integer ring.

2. The q-extension of the Hardy-Littlewood-type maximal operator

From (1.3) and the definition of q-Volkenborn integral, we first obtain the following theorem.

THEOREM 2.1. *Let μ_q be a strongly p-adic q-invariant distribution in the p-adic integer ring and $f \in C^{(1)}(\mathbb{Z}_p, \mathbb{C}_p)$. Then for any $r \in \mathbb{Z}$ and any $a \in \mathbb{Z}_p$, we have*

(1) $\int_{a+p^r\mathbb{Z}_p} q^{-p^r x} f(x) d\mu_{q^{p^r}}(x) = [p^r]_q \int_{\mathbb{Z}_p} q^{-x} f(a+p^r x) d\mu_q(x)$,

(2) $\int_{a+p^r\mathbb{Z}_p} d\mu_{q^{p^r}}(x) = q^{ap^r}[p^r]_q$.

Proof. (1) From the equation (1.3)and the extension of q-Volkenborn integral, we can derive the result as follow:

$$
\begin{aligned}
&\int_{a+p^r\mathbb{Z}_p} q^{-p^r x} f(x) d\mu_{q^{p^r}}(x) \\
&= \lim_{m\to\infty} \frac{1}{[p^{m+r}]_{q^{p^r}}} \sum_{x=0}^{p^m-1} q^{-p^r(a+p^r x)} f(a+p^r x) \left(q^{p^r}\right)^{a+p^r x} \\
&= \lim_{m\to\infty} \frac{1}{[p^{m+r}]_{q^{p^r}}} \sum_{x=0}^{p^m-1} f(a+p^r x) \\
&= [p^r]_q \lim_{m\to\infty} \frac{1}{[p^m]_q} \sum_{x=0}^{p^m-1} q^{-x} f(a+p^r x) q^x \\
&= [p^r]_q \int_{\mathbb{Z}_p} q^{-x} f(a+p^r x) d\mu_q(x).
\end{aligned}
$$

(2) By the same method of (1), we can obtain the following.

$$\begin{aligned}
&\int_{a+p^r\mathbb{Z}_p} d\mu_{q^{p^r}}(x) \\
&= \lim_{m\to\infty} \frac{1}{[p^{m+r}]_{q^{p^r}}} \sum_{x=0}^{p^m-1} \left(q^{p^r}\right)^{a+p^r x} \\
&= q^{ap^r} \lim_{m\to\infty} \frac{1}{[p^{m+r}]_{q^{p^r}}} \sum_{x=0}^{p^m-1} q^x \\
&= q^{ap^r}[p^r]_q \lim_{m\to\infty} \frac{1}{[p^m]_q} \sum_{x=0}^{p^m-1} q^x \\
&= q^{ap^r}[p^r]_q \int_{\mathbb{Z}_p} d\mu_q(x) \\
&= q^{ap^r}[p^r]_q.
\end{aligned}$$

□

Now, we define the q-extension of the Hardy-Littlewood-type maximal operator related to q-Volkenborn integral with a strong p-adic q-invariant distribution μ_q in the p-adic integer ring.

Definition 2.2. *Let μ_q be a strongly p-adic q-invariant distribution in the p-adic integer ring and $f \in C^{(1)}(\mathbb{Z}_p, \mathbb{C}_p)$. Then the q-extension of the Hardy-Littlewood-type maximal operator related to q-Volkenborn integral with a strong p-adic q-invariant distribution μ_q in the p-adic integer ring is defined by*

$$M_{p,q}f(a) = \sup_{r\in\mathbb{Z}} \frac{1}{\mu_{1,q^{p^r}}(a+p^r\mathbb{Z}_p)} \int_{a+p^r\mathbb{Z}_p} q^{-p^r x} f(x)\, d\mu_{q^{p^r}}(x), \tag{2.1}$$

for all $a \in \mathbb{Z}_p$.

We recall that the classical Hardy-Littlewood maximal operator M_μ is defined by

$$M_\mu f(a) = \sup_{a\in Q} \frac{1}{\mu(Q))} \int_Q |f(x)|\, d\mu(x), \tag{2.2}$$

where $f : \mathbb{R}^k \to \mathbb{R}^k$ is a locally bounded Lebesgue measurable function, μ is a Lebesgue measure on $(-\infty, \infty)$ and the supremum is taken over all cubes Q which are parallel to the coordinate axes. Note that the boundedness of the Hardy-Littlewood maximal operator serves as one of the most important tools used in the investigation of the properties of variable exponent spaces and operators acting on them(see [16,21]). The main idea of Definition 2.2 is to deal with the q-extension of the classical Hardy-Littlewood maximal operator in the space of p-adic Lipschitz functions on $\mathbb{Z}_p$ and to find the boundedness of them. From Theorem 2.1, we first obtain the following theorem.

THEOREM 2.3. *Let μ_q and $M_{p,q}$ be the same as in the definition 2.2. Then for any $f \in C^{(1)}(\mathbb{Z}_p, \mathbb{C}_p)$ and $x \in \mathbb{Z}_p$, we have*
(1) $M_{p,q}f(x) = \sup_{r\in\mathbb{Z}} \frac{1}{q^{p^r x}} \int_{\mathbb{Z}_p} q^{-z} f(x + p^r z) d\mu_q(z)$,
(2) $|M_{p,q}f(x)|_p \leq \sup_{r\in\mathbb{Z}} \frac{1}{|q^{xp^r}|_p} \|f\|_1 \|q^{-(\cdot)}\|_{L^1}$,
where $\|q^{-(\cdot)}\|_{L^1} = \int_{\mathbb{Z}_p} |q^{-z}|_p d\mu_q(z)$.

Proof. (1) From Theorem 2.3, we can derive the result as follows:

$$\begin{aligned} M_{p,q}f(x) &= \sup_{r\in\mathbb{Z}} \frac{1}{\mu_{1,q^{p^r}}(x+p^r\mathbb{Z}_p)} \int_{x+p^r\mathbb{Z}_p} q^{-p^r z} f(z) d\mu_{q^{p^r}}(z) \\ &= \sup_{r\in\mathbb{Z}} \frac{[p^r]_q}{q^{p^r x}[p^r]_q} \int_{\mathbb{Z}_p} q^{-z} f(x + p^r z) d\mu_q(z) \\ &= \sup_{r\in\mathbb{Z}} \frac{1}{q^{p^r x}} \int_{\mathbb{Z}_p} q^{-z} f(x + p^r z) d\mu_q(z). \end{aligned}$$

(2) From (1), we can obtain the following.

$$\begin{aligned} |M_{p,q}f(x)|_p &= \left| \sup_{r\in\mathbb{Z}} \frac{1}{\mu_{1,q^{p^r}}(x+p^r\mathbb{Z}_p)} \int_{x+p^r\mathbb{Z}_p} q^{-p^r z} f(z) d\mu_{q^{p^r}}(z) \right|_p \\ &\leq \sup_{r\in\mathbb{Z}} \left| \frac{1}{\mu_{1,q^{p^r}}(x+p^r\mathbb{Z}_p)} \int_{x+p^r\mathbb{Z}_p} q^{-p^r z} f(z) d\mu_{q^{p^r}}(z) \right|_p \\ &= \sup_{r\in\mathbb{Z}} \frac{1}{|q^{p^r x}|_p} \left| \int_{\mathbb{Z}_p} q^{-z} f(x + p^r z) d\mu_q(z) \right|_p \end{aligned}$$

$$\leq \sup_{r\in\mathbb{Z}} \frac{1}{|q^{p^r x}|_p} \int_{\mathbb{Z}_p} \left|q^{-z}\right|_p |f(x+p^r z)|_p d\mu_q(z)$$
$$\leq \sup_{r\in\mathbb{Z}} \frac{1}{|q^{p^r x}|_p} \int_{\mathbb{Z}_p} \left|q^{-z}\right|_p \|f\|_1 d\mu_q(z)$$
$$= \sup_{r\in\mathbb{Z}} \frac{1}{|q^{p^r x}|_p} \cdot \|f\|_1 \|q^{-(\cdot)}\|_{L^1}.$$

□

Note that Theorem 2.3(2) implies the supnorm-inequality for the q-extension of the Hardy-Littlewood-type maximal operator in the p-adic integer ring, in fact, Theorem 2.3(2) implies

$$\|M_{p,q}f\|_\infty = \sup_{x\in\mathbb{Z}_p} |M_{p,q}f(x)|_p \leq c\|q^{-(\cdot)}\|_{L^1}\|f\|_1 \tag{2.3}$$

where $c = \sup_{r\in\mathbb{Z}} \frac{1}{|q^{p^r x}|_p}$. By the equation 2.3, we can obtain the following corollary, which is the boundedness of the q-extension of the Hardy-Littlewood-type maximal operator in the p-adic integer ring.

Corollary 2.4. *Let μ_q and $M_{p,q}$ be the same as in the definition 2.2. Then $M_{p,q}$ is a bounded operator from $C^{(1)}(\mathbb{Z}_p, \mathbb{C}_p)$ into $L^\infty(\mathbb{Z}_p, \mathbb{C}_p)$, where $L^\infty(\mathbb{Z}_p, \mathbb{C}_p)$ is the space of all p-adic supnorm-bounded functions with the supnorm $\|h\|_\infty = \sup_{x\in\mathbb{Z}_p} |h(x)|_p$ for all $h \in L^\infty(\mathbb{Z}_p, \mathbb{C}_p)$.*

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*
Department of Mathematics and Computer Science
KonKuk University
Chungju 138-701, Republic of Korea
E-mail: leechae.jang@kku.ac.kr