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STABILITY OF GENERALIZED QUADRATIC MAPPINGS IN FUZZY NORMED SPACES ♮

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Abstract. In this paper we consider a generalized form of quadratic functional equations and establish new theorems about the generalized Hyers–Ulam stability of the generalized form of quadratic equations in fuzzy normed spaces.

1. Introduction

One of the interesting questions in the theory of functional analysis concerning the stability problem of functional equations is as follows: when is it true that a mapping satisfying approximately a functional equation must be close to an exact solution of the given functional equation? The first stability problem was raised by S. M. Ulam [16] during his talk at the University of Wisconsin in 1940. For very general functional equations, the concept of stability for functional equations arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. Thus the stability question of functional equations is that how do the solutions of the inequality differ from those of the given functional equation?

Following [2, 11], we present a fuzzy norm and a fuzzy normed space as follows.

Definition 1.1. Let X be a real linear spaces. A function $N : X \times \mathbb{R} \to [0,1]$(so-called fuzzy subset) is said to be a fuzzy norm on X if for all $x, y \in X$ and all $s, t \in \mathbb{R}$,

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- (N1) $N(x, c) = 0$ for $c \leq 0$;
- (N2) $x = 0$ if and only if $N(x, c) = 1$ for all $c > 0$;
- (N3) $N(cx, t) = N(x, \frac{t}{|c|})$ for $c \neq 0$;
- (N4) $N(x + y, s + t) \geq \min\{N(x, s), N(y, t)\}$;
- (N5) $N(x, \cdot)$ is a nondecreasing function on $\mathbb{R}$ and $\lim_{t\to\infty} N(x, t) = 1$.

The pair (X, N) is called a fuzzy normed linear space. Note that the fuzzy normed linear space (X, N) is exactly a Menger probabilistic normed linear space $(X, N, \triangle)$ when $\triangle := \min$. It is a locally convex first countable Hausdorff linear topological space [9, 15].

Let (X, N) be a fuzzy normed linear space. A sequence $\{x_n\}$ in X is said to be convergent to $x \in X$ if $\lim_{n\to\infty} N(x_n - x, t) = 1$ for all $t > 0$. In this case x is called the limit of the sequence $\{x_n\}$ and we denote it by $x = N - \lim_{n\to\infty} x_n$. A sequence $\{x_n\}$ in X is called Cauchy if for each $\varepsilon > 0$ and each $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $N(x_{n+p} - x_n, t) > 1 - \varepsilon$ for all $n \geq n_0$ and all $p > 0$.

It is known that every convergent sequence in a fuzzy normed linear space is Cauchy. The fuzzy normed linear space (X, N) is said to be complete if every Cauchy sequence in X is convergent and the fuzzy normed linear space (X, N) is called a fuzzy Banach space. For the various definitions, properties and continuity on a fuzzy normed space we refer to [3, 13, 14] for the reader.

Now we consider a mapping Q satisfying the following functional equation, which is introduced in [7],

$$\begin{aligned}(1.1)\qquad &\sum_{i=1}^{n} r_i Q\left(\sum_{j=1}^{n} r_j (x_i - x_j)\right) + \left(\sum_{i=1}^{n} r_i\right) Q\left(\sum_{i=1}^{n} r_i x_i\right) \\ &= \left(\sum_{i=1}^{n} r_i\right)^2 \sum_{i=1}^{n} r_i Q(x_i)\end{aligned}$$

for all vectors $x_1, \cdots, x_n$ and any fixed $r_1, \cdots, r_n \in (0, \infty)$, where $n \geq 2$ is a positive integer. As a special case, if $r_i = 1$ in (1.1) for all $i = 1, \cdots, n$, then the functional equation (1.1) reduces to

$$\sum_{i=1}^{n} Q\left(\sum_{j=1}^{n} (x_i - x_j)\right) + nQ\left(\sum_{i=1}^{n} x_i\right) = n^2 \sum_{i=1}^{n} Q(x_i),$$

which is exactly equivalent to the quadratic functional equation $Q(x + y) + Q(x - y) = 2Q(x) + 2Q(y)$ [6]. It is well-known [1] that a mapping Q

between real linear spaces satisfies the quadratic functional equation if and only if there is a unique symmetric biadditive mapping B such that $Q(x) = B(x, x)$ for all x. Let B be a symmetric biadditive mapping. Then the quadratic mapping Q given by $Q(x) := B(x, x)$ satisfies the equation (1.1). For this reason, the equation (1.1) is a generalized form of the quadratic functional equation. We observe that if a mapping $Q : X \to Y$ with $Q(0) = 0$ satisfies the equation (1.1) then $Q(L^k x) = L^{2k} Q(x)$ for any vector $x \in X$ and every integer $k \in \mathbb{Z}$, where $L := \sum_{i=1}^{n} r_i$.

Recently, fuzzy stability results concerning Cauchy, Jensen, quadratic and cubic functional equations in fuzzy normed linear spaces were investigated in [11, 13, 14]. In this paper we are going to investigate the generalized Hyers–Ulam stability problem that an approximate mapping satisfying approximately the functional equation (1.1) in fuzzy normed linear spaces can be approximated in a fuzzy sense by a true mapping satisfying exactly the equation (1.1). As a result, we obtain a better estimation than the result of Theorem 2.1 [12] for stability phenomenon in the fuzzy normed spaces. In Section 2, we study the generalized Hyers–Ulam stability problem using direct method by iteration. In Section 3, we investigate the generalized Hyers–Ulam stability problem using fixed point alternative by contraction mappings.

2. Stability of the equation (1.1) by direct method

For notational convenience, given a mapping $f : X \to Y$, we define the difference operator $Df : X^n \to Y$ of the equation (1.1) by

$$\begin{aligned} Df(x_1, \cdots, x_n) := &\sum_{i=1}^{n} r_i f\left(\sum_{j=1}^{n} r_j (x_i - x_j)\right) \\ &+ \left(\sum_{i=1}^{n} r_i\right) f\left(\sum_{i=1}^{n} r_i x_i\right) - \left(\sum_{i=1}^{n} r_i\right)^2 \sum_{i=1}^{n} r_i f(x_i) \end{aligned}$$

for all n-variables $x_1, \cdots, x_n \in X$, where $n \geq 2$ and $L := \sum_{i=1}^{n} r_i$. which acts as a perturbation of the equation (1.1). Throughout this section, we assume that X is a linear space, (Y, N) is a fuzzy Banach space and (Z, N') is a fuzzy normed space. Moreover, we assume that $N(x, \cdot)$ is a left continuous function on $\mathbb{R}$.

Theorem 2.1. *Assume that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality*

$$N(Df(x_1, x_2, \cdots, x_n), t) \geq N'(\varphi(x_1, \cdots, x_n), t) \tag{2.1}$$

and $\varphi : X^n \to Z$ is a mapping for which there is a constant $l \in \mathbb{R}$ satisfying $0 < |l| < L^2$ such that

$$N'(\varphi(Lx_1, \cdots, Lx_n), t) \geq N'(l\varphi(x_1, \cdots, x_n), t) \tag{2.2}$$

for all n-variables $x_1, \cdots, x_n \in X$, and $t > 0$. Then we can find a unique mapping $Q : X \to Y$ satisfying the equation $DQ(x_1, x_2, \cdots, x_n) = 0$ and the inequality

$$N(f(x) - Q(x), t) \geq N'\left(\frac{\varphi(x, \cdots, x)}{L(L^2 - |l|)}, t\right), \ t > 0 \tag{2.3}$$

for all $x \in X$.

Proof. We observe from (2.2) that

$$\begin{aligned} N'(\varphi(L^j x_1, \cdots, L^j x_n), t) &\geq N'(l^j \varphi(x_1, \cdots, x_n), t) \\ &= N'\left(\varphi(x_1, \cdots, x_n), \frac{t}{|l|^j}\right), \end{aligned}$$

$$N'(\varphi(L^j x_1, \cdots, L^j x_n), |l|^j t) \geq N'(\varphi(x_1, \cdots, x_n), t), \ t > 0 \tag{2.4}$$

for all $x_1, \cdots, x_n \in X$. Now, substituting x for $x_1, \cdots, x_n$ in the functional inequality (2.1), we obtain

$$\begin{aligned} &N(Lf(Lx) - L^3 f(x), t) \geq N'(\varphi(x, \cdots, x), t), \\ &\text{or,} \quad N\left(f(x) - \frac{f(Lx)}{L^2}, \frac{t}{L^3}\right) \geq N'(\varphi(x, \cdots, x), t) \end{aligned} \tag{2.5}$$

for all $x \in X$. Therefore it follows from (2.4), (2.5) with $L^j x$ in place of x, that

$$N\left(\frac{f(L^j x)}{L^{2j}} - \frac{f(L^{j+1} x)}{L^{2(j+1)}}, \frac{|l|^j t}{L^3 L^{2j}}\right) \geq N'(\varphi(x, \cdots, x), t)$$

for all $x \in X$ and any integer $j \geq 0$. So

$$\begin{aligned} N & \left(f(x) - \frac{f(L^k x)}{L^{2k}}, \sum_{j=0}^{k-1} \frac{|l|^j t}{L^3 L^{2j}} \right) \\ & = N \left(\sum_{j=0}^{k-1} \left(\frac{f(L^j x)}{L^{2j}} - \frac{f(L^{j+1} x)}{L^{2(j+1)}} \right), \sum_{j=0}^{k-1} \frac{|l|^j t}{L^3 L^{2j}} \right) \\ & \geq \min_{0 \leq j \leq k-1} \left\{ N \left(\frac{f(L^j x)}{L^{2j}} - \frac{f(L^{j+1} x)}{L^{2(j+1)}}, \frac{|l|^j t}{L^3 L^{2j}} \right) \right\} \\ & \geq N'(\varphi(x, \cdots, x), t), \ t > 0, \end{aligned}$$

which yields

$$\begin{aligned} N & \left(\frac{f(L^m x)}{L^{2m}} - \frac{f(L^{k+m} x)}{L^{2(k+m)}}, \sum_{j=0}^{k-1} \frac{|l|^{j+m} t}{L^3 L^{2(j+m)}} \right) \\ & \geq N'(\varphi(L^m x, \cdots, L^m x), |l|^m t) \\ & \geq N'(\varphi(x, \cdots, x), t), \ t > 0 \end{aligned} \tag{2.6}$$

for all $x \in X$ and any integers $k > 0, m \geq 0$. Hence one obtains

$$\begin{aligned} N & \left(\frac{f(L^m x)}{L^{2m}} - \frac{f(L^{k+m} x)}{L^{2(k+m)}}, t \right) \\ & \geq N' \left(\varphi(x, \cdots, x), \frac{t}{\sum_{j=0}^{k-1} \frac{|l|^{j+m}}{L^3 L^{2(j+m)}}} \right) \end{aligned} \tag{2.7}$$

for all $x \in X$ and any integers $k > 0, m \geq 0, t > 0$. Since $\sum_{j=0}^{\infty} \frac{|l|^j}{L^{2j}}$ is convergent series, we see by taking the limit $m \to \infty$ in the last inequality that a sequence $\left\{ \frac{f(L^k x)}{L^{2k}} \right\}$ is Cauchy in the fuzzy Banach space (Y, N) and so it converges in Y. Therefore a mapping $Q : X \to Y$ defined by

$$Q(x) := N - \lim_{k \to \infty} \frac{f(L^k x)}{L^{2k}}$$

is well defined for all $x \in X$. It means that $\lim_{k \to \infty} N \left(\frac{f(L^k x)}{L^{2k}} - Q(x), \ t \right) = 1, \ t > 0$. In addition, we see from (2.7) that

$$N \left(f(x) - \frac{f(L^k x)}{L^{2k}}, t \right) \geq N' \left(\varphi(x, \cdots, x), \frac{t}{\sum_{j=0}^{k-1} \frac{|l|^j}{L^3 L^{2j}}} \right), \tag{2.8}$$

and so

$$
\begin{aligned}
(2.9)\quad & N\left(f(x)-Q(x),t\right) \\
&\geq \min\left\{N\left(f(x)-\frac{f(L^kx)}{L^{2k}},(1-\varepsilon)t\right),N\left(\frac{f(L^kx)}{L^{2k}}-Q(x),\varepsilon t\right)\right\} \\
&\geq N'\left(\varphi(x,\cdots,x),\frac{t}{\sum_{j=0}^{k-1}\frac{|l|^j}{L^3L^{2j}}}\right), \\
&\geq N'\left(\varphi(x,\cdots,x),\varepsilon L(L^2-|l|)t\right),\ 0<\varepsilon<1,
\end{aligned}
$$

for sufficiently large k and for all $x\in X$, $t>0$. Since ε is arbitrary and N' is left continuous, we obtain

$$N\left(f(x)-Q(x),t\right)\geq N'\left(\varphi(x,\cdots,x),L(L^2-|l|)t\right),\ t>0$$

for all $x\in X$.

In addition it is clear from (2.7) and (N5) that the following relation

$$
\begin{aligned}
N\left(\frac{Df(L^kx_1,\cdots,L^kx_n)}{L^{2k}},t\right) &\geq N'\left(\varphi(L^kx_1,\cdots,L^kx_n),L^{2k}t\right) \\
&\geq N'\left(\varphi(x_1,\cdots,x_n),\frac{L^{2k}}{|l|^k}t\right) \\
&\to 1 \quad \text{as } k\to\infty
\end{aligned}
$$

holds for all $x_1,\cdots,x_n\in X$, $t>0$. Therefore, we obtain in view of $\lim_{k\to\infty}N\left(\frac{f(L^kx)}{L^{2k}}-Q(x),t\right)=1$ $(t>0)$,

$$
\begin{aligned}
& N(DQ(x_1,\cdots,x_n),t) \\
&\geq \min\Big\{N\Big(DQ(x_1,\cdots,x_n)-\frac{Df(L^kx_1,\cdots,L^kx_n)}{L^{2k}},\frac{t}{2}\Big), \\
&\qquad N\Big(\frac{Df(L^kx_1,\cdots,L^kx_n)}{L^{2k}},\frac{t}{2}\Big)\Big\} \\
&= N\Big(\frac{Df(L^kx_1,\cdots,L^kx_n)}{L^{2k}},\frac{t}{2}\Big) \text{ (for sufficiently large k)} \\
&\geq N'\Big(\varphi(x_1,\cdots,x_n),\frac{L^{2k}}{|l|^k}\frac{t}{2}\Big),\ t>0 \\
&\to 1 \quad \text{as } k\to\infty
\end{aligned}
$$

which implies $DQ(x_1,\cdots,x_n)=0$ by (N2). Thus we find that Q is a mapping satisfying the equation (1.1) and the inequality (2.3) near the approximate quadratic mapping $f:X\to Y$.

To prove the afore-mentioned uniqueness, we assume now that there is another mapping $Q' : X \to Y$ which satisfies the inequality (2.3). Then one establishes by the equality $Q'(L^k x) = L^{2k} Q'(x)$ and (2.3) that

$$\begin{aligned} N(Q(x) - Q'(x), t) &= N\left(\frac{Q(L^k x)}{L^{2k}} - \frac{Q'(L^k x)}{L^{2k}}, t\right) \\ &\geq \min\left\{N\left(\frac{Q(L^k x)}{L^{2k}} - \frac{f(L^k x)}{L^{2k}}, \frac{t}{2}\right), N\left(\frac{f(L^k x)}{L^{2k}} - \frac{Q'(L^k x)}{L^{2k}}, \frac{t}{2}\right)\right\} \\ &\geq N'\left(\varphi(L^k x, \cdots, L^k x), \frac{L(L^2 - |l|)L^{2k} t}{2}\right) \\ &\geq N'\left(\varphi(x, \cdots, x), \frac{L(L^2 - |l|)L^{2k} t}{2|l|^k}\right), \ t > 0, \forall k \in \mathbb{N} \end{aligned}$$

which tends to 1 as $k \to \infty$ by (N5). Therefore one obtains $Q(x) - Q'(x) = 0$ for all $x \in X$, completing the proof of uniqueness. □

THEOREM 2.2. *Assume that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality*

$$N(Df(x_1, x_2, \cdots, x_n), t) \geq N'(\varphi(x_1, \cdots, x_n), t) \tag{2.10}$$

and $\varphi : X^n \to Z$ is a mapping for which there is a constant $l \in \mathbb{R}$ satisfying $|l| > L^2$ such that

$$N'\left(\varphi(\frac{x_1}{L}, \cdots, \frac{x_n}{L}), t\right) \geq N'\left(\frac{1}{l}\varphi(x_1, \cdots, x_n), t\right) \tag{2.11}$$

for all n-variables $x_1, \cdots, x_n \in X$, and $t > 0$. Then we can find a unique mapping $Q : X \to Y$ satisfying the equation $DQ(x_1, x_2, \cdots, x_n) = 0$ and the inequality

$$N(f(x) - Q(x), t) \geq N'\left(\frac{\varphi(x, \cdots, x)}{L(|l| - L^2)}, t\right), \ t > 0 \tag{2.12}$$

for all $x \in X$.

Proof. It follows from (2.5) and (2.11) that

$$N\left(f(x) - L^2 f(\frac{x}{L}), \frac{t}{L|l|}\right) \geq N'(\varphi(x, \cdots, x), t), \ t > 0$$

for all $x \in X$. Therefore it follows that

$$N\left(f(x) - L^{2k} f(\frac{x}{L^k}), \sum_{j=0}^{k-1} \frac{L^{2j}}{L|l|^{j+1}} t\right) \geq N'(\varphi(x, \cdots, x), t), \ t > 0$$

for all $x \in X$ and any integers $k > 0$. Thus we see from the last inequality that

$$\begin{aligned} N\left(f(x) - L^{2k} f(\frac{x}{L^k}), t\right) &\geq N'\left(\varphi(x, \cdots, x), \frac{t}{\sum_{j=0}^{k-1} \frac{L^{2j}}{L|l|^{j+1}}}\right) \\ &\geq N'(\varphi(x, \cdots, x), L(|l| - L^2)t),\ t > 0. \end{aligned}$$

The remaining assertion goes through by the similar way to corresponding part of Theorem 2.1. □

We obtain the following corollaries concerning the stability for approximate mappings controlled by a sum of powers of norms.

Corollary 2.3. *Let X be a normed space and $(\mathbb{R}, N')$ a fuzzy normed space. Assume that there exist real numbers $\theta \geq 0$ and $p \in \mathbb{R} \setminus \{2\}$ such that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality*

$$N(Df(x_1, x_2, \cdots, x_n), t) \geq N'(\theta(\sum_{i=1}^{n} \|x_i\|^p), t)$$

for all n-variables $x_1, \cdots, x_n \in X$, and $t > 0$. Then we can find a unique mapping $Q : X \to Y$ satisfying the equation $DQ(x_1, x_2, \cdots, x_n) = 0$ and the inequality

$$N(f(x) - Q(x), t) \geq N'\left(\frac{n\theta\|x\|^p}{L(|L^p - L^2|)}, t\right),\ t > 0$$

for all $x \in X$.

We obtain the following corollaries concerning the stability for approximate mappings controlled by a product of powers of norms.

Corollary 2.4. *Let X be a normed space and $(\mathbb{R}, N')$ a fuzzy normed space. Assume that there exist real numbers $\theta \geq 0$ and $p_i \in \mathbb{R}$ with $p := \sum_{i=1}^{n} p_i \neq 2$ such that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality*

$$N(Df(x_1, x_2, \cdots, x_n), t) \geq N'(\theta \prod_{i=1}^{n} \|x_i\|^{p_i}, t)$$

for all n-variables $x_1, \cdots, x_n \in X$, and $t > 0$. Then we can find a unique mapping $Q : X \to Y$ satisfying the equation $DQ(x_1, x_2, \cdots, x_n) = 0$ and the inequality

$$N(f(x) - Q(x), t) \geq N'\left(\frac{\theta\|x\|^p}{L(|L^p - L^2|)}, t\right),\ t > 0$$

for all $x \in X$.

Corollary 2.5. *Assume that there exist a real number* $\theta \geq 0$ *such that a mapping* $f : X \to Y$ *with* $f(0) = 0$ *satisfies the inequality*

$$N(Df(x_1, x_2, \cdots, x_n), t) \geq N'(\theta, t)$$

for all n*-variables* $x_1, \cdots, x_n \in X$, *and* $t > 0$. *Then we can find a unique mapping* $Q : X \to Y$ *satisfying the equation* $DQ(x_1, x_2, \cdots, x_n) = 0$ *and the inequality*

$$N(f(x) - Q(x), t) \geq N'\left(\frac{\theta}{L(L^2-1)}, t\right), \ t > 0$$

for all $x \in X$.

We remark that if $\theta = 0$, then $N(Df(x_1, x_2, \cdots, x_n), t) \geq N(0, t) = 1$ and so $Df(x_1, x_2, \cdots, x_n) = 0$. Thus we get $f = Q$ is itself a quadratic mapping.

3. Stability of the equation (1.1) by fixed point method

Now, in the next theorem we are going to consider a stability problem concerning the stability of the equation (1.1) by using a fixed point theorem of the alternative for contractions on a generalized complete metric space due to B. Margolis and J.B. Diaz [8].

Theorem 3.1. *Assume that there exist constants* $l \in \mathbb{R}$ *and* $q > 0$ *satisfying* $0 < |l|^{\frac{1}{q}} < L^2$ *such that a mapping* $f : X \to Y$ *with* $f(0) = 0$ *satisfies the inequality*

$$N\left(Df(x_1, x_2, \cdots, x_n), \sum_{i=1}^{n} t_i\right) \geq \min_{1 \leq i \leq n}\{N'(\varphi(x_i), t_i^q)\} \tag{3.1}$$

for all $x_i \in X$, *and* $t_i > 0$ $(i = 1, \cdots, n)$ *and* $\varphi : X \to Z$ *is a mapping satisfying*

$$\varphi(Lx) = l\varphi(x) \tag{3.2}$$

for all $x \in X$. *Then there exists a unique mapping* $Q : X \to Y$ *satisfying the equation* $DQ(x_1, x_2, \cdots, x_n) = 0$ *and the inequality*

$$N(f(x) - Q(x), t) \geq N'\left(\frac{n^q}{L^q(L^2 - |l|^{\frac{1}{q}})^q}\varphi(x), \ t\right), \ t > 0 \tag{3.3}$$

for all $x \in X$. *Furthermore, if a mapping* $r \to f(rx)$ *is continuous in* $r \in \mathbb{R}$ *for each fixed* $x \in X$, *then* $Q(rx) = r^2 Q(x)$ *for all* $r \in \mathbb{R}$.

Proof. We consider the set

$$\Omega := \{g : X \to Y|\ g(0) = 0\}$$

and define a generalized metric on Ω as follows

$$d_\Omega(g,h) := \inf\Big\{K \in (0,\infty] \mid N(g(x)-h(x),Kt) \geq N'(\varphi(x),t^q),\ \forall\ x \in X, t > 0\Big\}.$$

Then one can easily see that (Ω, d_Ω) is a complete generalized metric space [5, 10].

Now, we define an operator $J : \Omega \to \Omega$ as

$$Jg(x) = \frac{g(Lx)}{L^2},$$

for all $g \in \Omega, x \in X$.

We first prove that J is strictly contractive on Ω. For any $g, h \in \Omega$, let $\varepsilon \in [0,\infty]$ be any constant with $d_\Omega(g,h) \leq \varepsilon$. Then we deduce from use of (3.2) that if $d_\Omega(g,h) \leq \varepsilon$,

$$\Longrightarrow N(g(x)-h(x),\varepsilon t) \geq N'(\varphi(x),t^q),$$

$$\Longrightarrow N\left(\frac{g(Lx)}{L^2} - \frac{h(Lx)}{L^2}, \frac{|l|^{\frac{1}{q}}}{L^2}\varepsilon t\right) \geq N'(\varphi(Lx),|l|t^q) = N'(\varphi(x),t^q),$$

$$\Longrightarrow N(Jg(x)-Jh(x), \frac{|l|^{\frac{1}{q}}}{L^2}\varepsilon t) \geq N'(\varphi(x),t^q),\ \forall\ x \in X, t > 0,$$

$$\Longrightarrow d_\Omega(Jg,Jh) \leq \frac{|l|^{\frac{1}{q}}}{L^2}\varepsilon.$$

Since ε is arbitrary constant with $d_\Omega(g,h) \leq \varepsilon$, we see that for any $g, h \in \Omega$,

$$d_\Omega(Jg,Jh) \leq \frac{|l|^{\frac{1}{q}}}{L^2} d_\Omega(g,h),$$

which implies J is strictly contractive with constant $\frac{|l|^{\frac{1}{q}}}{L^2} < 1$ on Ω.

We now want to show that $d(f,Jf) < \infty$. If we put $x_i := x, t_i := t(i = 1,\cdots,n)$ in (3.1), then we arrive at

$$N\left(f(x) - \frac{f(Lx)}{L^2}, \frac{n}{L^3}t\right) \ \geq \ N'(\varphi(x),t^q),$$

which yields $d_\Omega(f,Jf) \leq \frac{n}{L^3}$, and so $d_\Omega(J^kf,J^{k+1}f) \leq d_\Omega(f,Jf) \leq \frac{n}{L^3}$ for all $k \in \mathbb{N}$.

Using the fixed point theorem of the alternative for contractions on a generalized complete metric space due to B. Margolis and J.B. Diaz [8],

(i) we see that there is a mapping $Q : X \to Y$ with $Q(0) = 0$ such that

$$d_\Omega(f, Q) \leq \frac{1}{1 - \frac{|l|}{L^2}} d_\Omega(f, Jf) \leq \frac{n}{L(L^2 - |l|^{\frac{1}{q}})}$$

and Q is a fixed point of the operator J, that is, $\frac{1}{L^2}Q(Lx) = JQ(x) = Q(x)$ for all $x \in X$. Thus we can get

$$N\left(f(x) - Q(x), \frac{n}{L(L^2 - |l|^{\frac{1}{q}})}t\right) \geq N'(\varphi(x), t^q),$$

$$N(f(x) - Q(x), t) \geq N'\left(\varphi(x), \left(\frac{L(L^2 - |l|^{\frac{1}{q}})}{n}\right)^q t^q\right)$$

for all $t > 0$ and all $x \in X$;

(ii) we find that $d_\Omega(J^k f, Q) \to 0$ as $k \to \infty$. Thus we obtain

$$\begin{aligned} N\left(\frac{f(L^k x)}{L^{2k}} - Q(x), t\right) &= N\left(f(L^k x) - Q(L^k x), L^{2k} t\right) \\ &\geq N'\left(\frac{n^q \varphi(L^k x)}{L^q (L^2 - |l|^{\frac{1}{q}})^q}, L^{2qk} t^q\right) \\ &= N'\left(\frac{n^q \varphi(x)}{L^q (L^2 - |l|^{\frac{1}{q}})^q}, \left(\frac{L^{2q}}{|l|}\right)^k t^q\right) \\ &\to 1 \quad \text{as} \quad k \to \infty, \ \left(\frac{L^{2q}}{|l|} > 1\right) \end{aligned}$$

for all $t > 0$ and all $x \in X$, that is,

$$N - \lim_{k\to\infty} \frac{f(L^k x)}{L^{2k}} = Q(x)$$

for all $x \in X$. In addition, it follows from the condition (3.2) and (N4) that

$$\begin{aligned} N\left(\frac{Df(L^k x_1, \cdots, L^k x_n)}{L^{2k}}, t\right) &\geq \min_{1\leq i\leq n}\left\{N'\Big(\varphi(L^k x_i), \frac{L^{2qk} t^q}{n^q}\Big)\right\} \\ &= \min_{1\leq i\leq n}\left\{N'\Big(|l|^k \varphi(x_i), \frac{L^{2qk} t^q}{n^q}\Big)\right\} \end{aligned} \tag{3.4}$$

$$
\begin{aligned}
&= \min_{1\le i\le n}\Big\{N'\Big(\varphi(x_i), \Big(\frac{L^{2q}}{|l|}\Big)^k \frac{t^q}{n^q}\Big)\Big\} \\
&\to 1 \quad \text{as} \quad k\to\infty,\ t>0
\end{aligned}
$$

for all $x \in X$. Therefore we obtain by way of (N4) and (3.4)

$$
\begin{aligned}
&N(DQ(x_1, \cdots, x_n), t) \\
&\ge \min\Big\{N\Big(DQ(x_1,\cdots,x_n) - \frac{Df(L^kx_1,\cdots,L^kx_n)}{L^{2k}}, \frac{t}{2}\Big), \\
&\qquad N\Big(\frac{Df(L^kx_1,\cdots,L^kx_n)}{L^{2k}}, \frac{t}{2}\Big)\Big\} \\
&= N\Big(\frac{Df(L^kx_1,\cdots,L^kx_n)}{L^{2k}}, \frac{t}{2}\Big) \text{ (for sufficiently large k)} \\
&= \min_{1\le i\le n}\Big\{N'\Big(\varphi(x_i), \Big(\frac{L^{2q}}{|l|}\Big)^k \frac{t^q}{2^q n^q}\Big)\Big\} \\
&\to 1 \quad \text{as } k\to\infty,\ t>0,
\end{aligned}
$$

which implies $DQ(x_1, \cdots, x_n) = 0$ by (N2) and so the mapping Q satisfies the equation (1.1).

(iii) we know that the mapping Q is a unique fixed point of the operator J in the set $\Delta = \{g \in \Omega |\ d_\Omega(f, g) < \infty\}$. Thus if we assume that there exists another Euler–Lagrange type quadratic mapping $q : X \to Y$ satisfying the inequality (3.3) then

$$
q(x) = \frac{q(Lx)}{L^2} = Jq(x)
$$

and so q is a fixed point of the operator J. In view of (3.3) and the definition of d_Ω, we deduce that

$$
d_\Omega(f, q) \le \frac{n}{L(L^2 - |l|^{\frac{1}{q}})} < \infty,
$$

viz., $q \in \Delta = \{g \in \Omega |\ d_\Omega(f, g) < \infty\}$. By the uniqueness of the fixed point of J in Δ, we find that $Q = q$, which proves the uniqueness of Q satisfying the inequality (3.3). This ends the proof of the theorem. □

We remark that if $n = 2, r_i = 1 (i = 1, 2), L = 2 = l, q > \frac{1}{2}$ and $\varphi(x) = x$ in a special case, then the estimation (3.3) is better than that of Theorem 2.1 in the paper [12].

Theorem 3.2. *Assume that there exist constants $l \in \mathbb{R}$ and $q > 0$ satisfying $0 < |l|^{\frac{1}{q}} > L^2$ such that a mapping $f : X \to Y$ with $f(0) = 0$*

satisfies the inequality

$$N\left(Df(x_1,x_2,\cdots,x_n),\sum_{i=1}^{n}t_i\right)\geq\min_{1\leq i\leq n}\{N'(\varphi(x_i),t_i^q)\}$$

for all $x_i \in X$, *and* $t_i > 0$ $(i = 1, \cdots, n)$ *and* $\varphi : X \to Z$ *is a mapping satisfying*

$$\varphi(Lx) = l\varphi(x)$$

for all $x \in X$. *Then there exists a unique mapping* $Q : X \to Y$ *satisfying the equation* $DQ(x_1, x_2, \cdots, x_n) = 0$ *and the inequality*

$$N(f(x)-Q(x),t)\geq N'\left(\frac{n^q}{L^q(|l|^{\frac{1}{q}}-L^2)^q}\varphi(x),\ t^q\right),\ t>0$$

for all $x \in X$. *Furthermore, if a mapping* $r \to f(rx)$ *is continuous in* $r \in \mathbb{R}$ *for each fixed* $x \in X$, *then* $Q(rx) = r^2Q(x)$ *for all* $r \in \mathbb{R}$.

Proof. The proof of this theorem is similar to that of Theorem 3.1. □

4. Conclusion

It is important to consider the problem to study the best possible estimation of the difference $f(x) - Q(x)$ in stability problem of nonlinear functional equations [4]. We therefore establish the generalized Hyers–Ulam stability of the functional equation (1.1) in the fuzzy normed spaces and thus we provide an improved possible estimation of the difference $f(x) - Q(x)$ in stability problem of quadratic functional equations.

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