

PERIODIC POINTS WHOSE STABLE SETS HAVE NONEMPTY INTERIOR

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ABSTRACT. In this paper, we show that if a homeomorphism has the pseudo-orbit-tracing-property and its nonwandering set is locally connected, then the points whose stable sets have nonempty interior are periodic points.

1. Introduction and preliminaries

Throughout this paper, let X be a compact metric space with a metric function d and f be a homeomorphism of X . Our purpose here is to study dynamical properties of homeomorphisms together with the related concepts of nonwanderingness, chain recurrence and the pseudo-orbit-tracing-property. In [7], Ruess and Summers studied the motions whose limit sets consist of a single periodic motion. In [6], Ombach gave necessary and sufficient conditions that a limit set of a point consists of a single periodic orbit under the condition that f is expansive homomorphism with the pseudo-orbit-tracing-property. Also, author studied stable points whose limit sets consist of single periodic orbit [4] and also study the dynamical properties of nonwandering points whose stable sets have nonempty interior [5].

Here, we show that if f has the pseudo-orbit-tracing-property and its nonwandering set is locally connected, then the points whose stable sets have nonempty interior are periodic points.

For x in X , $O_f(x)$, $O_f^+(x)$ and $O_f^-(x)$ denote the f -orbit, positive f -orbit and negative f -orbit of x , respectively. Let $C(f)$ and $\Omega(f)$ be the *recurrent set* and *nonwandering set* of f , respectively. Recall that $C(f) = \{x \in X : x \in \omega_f(x) \cap \alpha_f(x)\}$, where $\omega_f(x)$ and $\alpha_f(x)$ denote the positive and negative limit set of x for f , respectively, and $\Omega(f) = \{x \in X : \text{for every}$

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neighborhood U of x and integer $n_0 > 0$ there exists an integer $n \geq n_0$ such that $f^n(U) \cap U \neq \emptyset$.

A sequence of points $\{x_i\}_{i \in (a,b)}$, $(-\infty \leq a < b \leq \infty)$, is called a δ -pseudo-orbit of f if $d(f(x_i), x_{i+1}) < \delta$ for $i \in (a, b-1)$. A finite pseudo-orbit $\{x_0, x_1, \dots, x_n\}$ is called a pseudo-orbit from x_0 to x_n . Let $x, y \in X$. x is related to y (written $x \sim y$) if there are γ -pseudo-orbits of f from x to y and y to x for every $\gamma > 0$. $CR(f) = \{x \in X : x \sim x\}$ is called the *chain recurrent set* of f . The relation $\sim$ is an equivalence relation in $CR(f)$. A *chain component* is an equivalence class in $CR(f)$ under the relation $\sim$. A sequence of points $\{x_i\}_{i \in (a,b)}$ is called ε -traced by $x \in X$ if $d(f^i(x), x_i) < \varepsilon$ holds for $i \in (a, b)$. We say that f has the *pseudo-orbit-tracing-property* if for every $\varepsilon > 0$ there is a $\delta > 0$ such that every δ -pseudo-orbit of f can be ε -traced by some point $x \in X$.

A subset M of X is called *f-minimal* if f -orbit of every point in M is dense in M .

Let $B(x, \varepsilon)$ denote $\{y \in X : d(x, y) < \varepsilon\}$ and $\overline{M}$ denote the closure of $M \subset X$.

2. Basic results

Here, we introduce several lemmas which are used in this paper.

LEMMA 1 [4]. *Each connected component of $CR(f)$ is contained in a chain component of $CR(f)$.*

LEMMA 2 [2,3]. *If f has the pseudo-orbit-tracing-property, then the following properties hold;*

- (1) f^k has the pseudo-orbit-tracing-property for every nonzero integer k ;
- (2) f restricted to its nonwandering set has the pseudo-orbit-tracing-property;
- (3) if Y is an open and closed f -invariant subset of X , then f restricted to Y has the pseudo-orbit-tracing-property;
- (4) $\overline{C(f)} = \Omega(f) = CR(f)$ holds.

LEMMA 3 [1]. *If X is a nontrivial connected f -minimal set, then f does not satisfy the pseudo-orbit-tracing-property.*

From here to the end of this paper, we assume that f has the pseudo-orbit-tracing-property and its nonwandering set is locally connected. Note that if the nonwandering set is connected, then f must be a nonwandering homeomorphism.

THEOREM 4. *There is a decomposition of $\Omega(f)$ satisfying the followings;*

- (1) *There is a decomposition of $\Omega(f)$ into disjoint closed sets ; $\Omega(f) = \Omega_1 \cup \Omega_2 \cup \cdots \cup \Omega_k$ such that each Ω_i is f -invariant and f restricted to each Ω_i is topologically transitive.*
- (2) *Again, there is a decomposition of each Ω_i into disjoint closed sets ; $\Omega_i = \Omega_1^i \cup \Omega_2^i \cup \cdots \cup \Omega_{n_i}^i$ and these sets are permuted by f .*

Here, each Ω_i is a chain component and each Ω_j^i is a connected component of $\Omega(f)$

Proof. By the local connectedness of $\Omega(f)$, we can find a finite number of pairwise disjoint connected components of $\Omega(f)$ which form a covering of $\Omega(f)$. According to Lemma 1, $\Omega(f)$ decomposes as a finite disjoint union of chain components of $\Omega(f)$:

$$\Omega(f) = \Omega_1 \cup \Omega_2 \cup \cdots \cup \Omega_k,$$

and each Ω_i decomposes as a finite disjoint union of connected components of $\Omega(f)$:

$$\Omega_i = \Omega_1^i \cup \Omega_2^i \cup \cdots \cup \Omega_{n_i}^i.$$

It is well known that each chain component is f -invariant closed. It is not difficult to show that f restricted to each Ω_i is topologically transitive.

Next, we prove the part (2) of this result. Fix $i \in \{1, 2, \dots, k\}$. Observe that since continuous images of connected sets are also connected, we have

$$(1) \quad f(\Omega_{j_1}^i) \cap \Omega_{j_2}^i \neq \emptyset \quad \text{implies} \quad f(\Omega_{j_1}^i) = \Omega_{j_2}^i.$$

Using this fact, now, we prove that $\{\Omega_j^i\}$, $(1 \leq j \leq n_i)$, is permuted by f . Since $f|_{\Omega_i}$ is topologically transitive, there is a point in Ω_1^i whose orbit is dense in Ω_i . In view of (1), this implies that, for every $1 \leq j \leq n_i$,

$$(2) \quad f^{n_i}(\Omega_1^i) = \Omega_j^i \quad \text{for some integer } n_j.$$

Without loss of generality, let us assume that $\{\Omega_1^i, \Omega_2^i, \dots, \Omega_l^i\}$, $(1 \leq l \leq n_i)$, is permuted by f . Then, for j with $l < j \leq n_i$, we have

$$(3) \quad f^n(\Omega_1^i) \cap \Omega_j^i = \emptyset \quad \text{for every integer } n.$$

The contradiction between (2) and (3) shows that $\{\Omega_j^i\}$ is permuted by f . By renumbering suitably, we get a decomposition of Ω_i , as given in the part (2) of this theorem. This completes the proof. $\square$

For $x \in X$ the *stable set* and *unstable set* of a homeomorphism f are defined by

$$W^s(x, f) = \{y \in X : \lim_{n \rightarrow \infty} d(f^n(x), f^n(y)) = 0\},$$

$$W^u(x, f) = \{y \in X : \lim_{n \rightarrow \infty} d(f^{-n}(x), f^{-n}(y)) = 0\}.$$

Let $\text{int}W^\sigma(x, f)$, $(\sigma = s, u)$ denote the set of interior points of $W^\sigma(x, f)$ and let $\text{int}W^\sigma(x, f)_{\Omega(f)}$ denote the set of interior points of $W^\sigma(x, f) \cap \Omega(f)$ in the subspace $\Omega(f)$ of X .

LEMMA 5. *If K is a nontrivial connected component of $\Omega(f)$ and $x \in K$, then $\text{int}W^\sigma(x, f) = \emptyset$, $(\sigma = s, u)$.*

Proof. Let K be a nontrivial connected component of $\Omega(f)$ and $x \in K$. We start with $\sigma = s$. Suppose, on the contrary that $\text{int}W^s(x, f) \neq \emptyset$. By the Theorem 4, there is a positive integer m such that $f^m(K) = K$. Let g denote f^m for convenience and $0 < c_0 < \min\{d(K_i, K_j)\}$, where $\{(K_i, K_j)\}$ is the set of pairs of disjoint connected component of $\Omega(f)$. Note that $K \subset \Omega(g)$. It is clear that also $\text{int}W^s(x, g)$ is not an empty set.

First, we claim that $\omega_g(x) = K$. To see this, assume on the contrary, that $\omega_g(x)$ is a proper subset of K . Let $p \in K \setminus \omega_g(x)$ and $d(p, \omega_g(x)) = \alpha$,

since $\text{int}W^s(x, g)_{\Omega(g)} \neq \emptyset$, we can take a point y and a positive number ξ with $\xi < \min\{c_0, \alpha/4\}$ satisfying that

$$y \in B(y, \xi) \cap \Omega(g) \subset \text{int}W^s(x, g)_{\Omega(g)}.$$

Let $\delta = \delta(\xi)$ ($< \xi$) be a positive number with the property of the pseudo-orbit-tracing-property of g . Since K is connected, we can take a δ -pseudo-orbit $\{z_i\} = \{z_1, z_2, \dots, z_N\}$ of g from p to y . By the pseudo-orbit-tracing-property of g , there exists a point p_1 ξ -tracing this pseudo-orbit $\{z_i\}$. In particular, we get

$$p_1 \in B(p, \xi) \quad \text{and} \quad g^N(p_1) \in B(y, \xi).$$

Using the continuity of g , we can choose an open neighborhood W of p_1 satisfying that

$$p_1 \in W \subset B(p, \xi) \quad \text{and} \quad g^N(W) \subset B(y, \xi).$$

On the other hand, since $\overline{C(g)} = \Omega(g)$, we can choose a point p_2 satisfying that

$$p_2 \in W \cap C(g) \quad \text{and} \quad g^N(p_2) \in B(y, \xi) \subset \text{int}W^s(x, g)_{\Omega(g)}.$$

Since $d(g^n(g^N(p_2)), g^n(x))$ tends to zero as n goes to infinity, there exists a positive integer L_1 satisfying that

$$d(g^n(g^N(p_2)), g^n(x)) < \frac{\alpha}{4}, \quad \text{for every } n > L_1.$$

Also, there is a positive integer L_2 satisfying that

$$d(g^n(x), \omega_g(x)) < \frac{\alpha}{4} \quad \text{for every } n > L_2.$$

Let L_3 be a positive integer with $L_3 > \max\{L_1, L_2\}$. Then we have

$$(1) \quad d(\omega_g(x), g^n(g^N(p_2))) < d(\omega_g(x), g^n(x)) + d(g^n(x), g^n(g^N(p_2))) < \frac{\alpha}{2}.$$

for every $n > L_3$. Since $p_2 \in C(g)$, we can select a positive integer L with $L > L_3$ satisfying that

$$(2) \quad g^L(g^N(p_2)) \in W \subset B(p, \xi) \subset B(p, \frac{\alpha}{4}).$$

By (1) and (2), we get the following inclusions.

$$g^L(g^N(p_2)) \subset W \cap B(\omega_g(x), \frac{\alpha}{2}) \subset B(p, \frac{\alpha}{2}) \cap B(\omega_g(x), \frac{\alpha}{2}).$$

This shows that $d(p, \omega_g(x)) < \alpha$. This contradicts the fact that $d(p, \omega_g(x)) = \alpha$. Hence, we obtain that $\omega_g(x) = K$ as desired.

Since restriction $g|_K$ has the pseudo-orbit-tracing-property and K is connected, K is not g -minimal by Lemma 3. Let M be a g -minimal proper subset of K . Let $q \in K \setminus M$ and $d(q, M) = \beta$. Choose $\varepsilon > 0$ with $\varepsilon < \min\{c_0, \beta/3\}$. Choose a point y and a positive number $\zeta < \varepsilon$ such that

$$B(y, \zeta) \cap \Omega(g) \subset \text{int}W^s(x, g)_{\Omega(g)}.$$

Let $\delta = \delta(\zeta)$ be a number with the property of the pseudo-orbit-tracing-property of g . Let $z \in M$. Take a δ -pseudo-orbit $\{v_0, v_1, \dots, v_L\}$ of g from y to z . Consider the following δ pseudo-orbit of g ;

$$\{b_i\}_{i=0}^\infty = \{v_0, v_1, \dots, v_L, g(z), g^2(z), g^3(z), \dots\}.$$

Then there exists a point y_b ζ -tracing the pseudo-orbit $\{b_i\}$. In particular, we have

$$(3) \quad d(y, y_b) < \zeta \quad \text{and} \quad d(g^i(g^L(y_b)), g^i(z)) < \zeta$$

for every $i \geq 0$. Note that

$$(4) \quad M = \overline{O_g^+(g^j(z))} \quad \text{for every integer } j.$$

Therefore, by (3) and (4), the following inclusions hold ;

$$(5) \quad \omega_g(y_b) = \omega_g(g^L(y_b)) \subset \overline{B(M, \zeta)} \subset \overline{B(M, 2\zeta)}.$$

Also, $y_b \in B(y, \zeta) \cap \Omega(g) \subset \text{int}W^s(x, g)_{\Omega(g)}$ implies that there is a positive integer J satisfying that

$$d(g^n(x), g^n(y_b)) < \zeta \quad \text{for every } n \geq J.$$

This implies that

$$(6) \quad \omega_g(x) = \omega_g(g^J(x)) \subset \overline{B(\omega_g(g^J(y_b)), \zeta)} = \overline{B(\omega_g(y_b), \varepsilon)}.$$

By (5) and (6), we obtain the following inclusions ;

$$K = \omega_g(x) \subset \overline{B(\omega_g(y_b), \varepsilon)} \subset \overline{B(M, 2\zeta + \varepsilon)} \subset B(M, 3\varepsilon).$$

Therefore, we have $d(q, M) < 3\varepsilon < \beta$ and this contradicts the fact that $d(q, M) = \beta$. Hence, we conclude that $\text{int}W^s(x, f) = \emptyset$. One can use the similar method used in the above argument to obtain the fact that $\text{int}W^u(x, f) = \emptyset$. This completes the proof. $\square$

THEOREM 2.6. *Let $\text{int}W^\sigma(x, f) \neq \emptyset$, ($\sigma = s, u$). Then the followings hold :*

- (1) *If x is an wandering point of f , then its limit set consists of single periodic orbit.*
- (2) *If x is a nonwandering point of f , then x is a periodic point.*

Proof. (1). See [5].

(2). Let x be a nonwandering point and $\sigma = s$. If x is in a nontrivial connected component of $\Omega(f)$ Then, by the previous lemma, $\text{int}W^s(x, f) = \emptyset$. Thus x must be in a trivial connected component K of $\Omega(f)$. Therefore $\{x\} = K$ and by Theorem 4, $f^n(x) = x$ for some nonnegative integer n . Thus x is a periodic point. The conclusion in the case that $\sigma = u$ is also obtained similarly. $\square$

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