JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **23**, No. 4, December 2010

GALOIS ACTIONS OF A CLASS INVARIANT OVER QUADRATIC NUMBER FIELDS WITH DISCRIMINANT $D \equiv -3 \pmod{36}$

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ABSTRACT. A class invariant is the value of a modular function that generates a ring class field of an imaginary quadratic number field such as the singular moduli of level 1. In this paper, using Shimura Reciprocity Law, we compute the Galois actions of a class invariant from a generalized Weber function $\mathfrak{g}_2$ over quadratic number fields with discriminant $D \equiv -3 \pmod{36}$.

1. Introduction

Let K be an imaginary quadratic number field with the discriminant D with ring of integer $\mathcal{O} = \mathbb{Z}[\theta]$ where

$$(1.1) \qquad \theta := \begin{cases} \frac{\sqrt{D}}{2}, & \text{if } D \equiv 0 \pmod{4}; \\ \frac{-1+\sqrt{D}}{2}, & \text{if } D \equiv 1 \pmod{4}. \end{cases}$$

Then the theory of complex multiplication states that the modular invariant $j(\mathcal{O}) = j(\theta)$ generates the *ring class field* $H_{\mathcal{O}}$ over K with degree $[H_{\mathcal{O}} : K] = h(\mathcal{O})$, the class number of $\mathcal{O}$, and the conjugates of $j(\theta)$ under the action of $Gal(H_{\mathcal{O}}/K)$ are *singular moduli* $j(\tau)$, where $\tau := \tau_Q$ is the Heegner point determined by $Q(\tau_Q, 1) = 0$ for a positive definite integral primitive binary quadratic forms

$$Q(x, y) = [a, b, c] = ax^2 + bxy + cy^2$$

with discriminant $D = b^2 - 4ac$.

Received October 20, 2010; Accepted November 24, 2010.
2010 Mathematics Subject Classification: Primary 11F37; Secondary 11F30.
Key words and phrases: class invariants, Galois actions, Weber functions.
This work was supported by the Korea Science and Engineering Foundation(KOSEF) grant funded by the Korea government(MEST)(No. 2010-0000607).

In his Lehrbuch der Algebra [8], H. Weber calls the value of a modular function $f(\theta)$ a *class invariant* if we have

$$K(f(\theta)) = K(j(\theta)).$$

Despite a long history of the problem, one began to treat class invariants in a systemic and algorithmic way only after Shimura Reciprocity Law [6] became available. The reciprocity law provides not only a method of systematically determining whether $f(\theta)$ is a class invariant but also a description of the Galois conjugates of $f(\theta)$ under $Gal(H_{\mathcal{O}}/K)$. This tool is well illustrated in several works by Reinier M. Bröker, Alice Gee, and Peter Stevenhagen in [1, 3, 4, 5, 7]. Bröker's Ph. D thesis [1] discusses p-adic theory of class invariants as well.

Gee determined the class invariants from a generalized Weber function $\mathfrak{g}_2$ by using the Shimura Reciprocity Law as follows:

THEOREM 1.1. [4, p.73, Theorem 1] *Let K be an imaginary quadratic number field of discriminant $D \equiv -3 \pmod{36}$ with the ring of integer $\mathcal{O} = \mathbb{Z}[\theta]$. Suppose $\theta = \frac{-B+\sqrt{D}}{2}$ as defined in (1.1). Then $\frac{1}{\sqrt{-3}}\mathfrak{g}_2^2(\theta)$ gives an integral generator for $H_{\mathcal{O}}$ over K.*

In this paper, we compute the Galois actions of the class invariant $\frac{1}{\sqrt{-3}}\mathfrak{g}_2^2(\theta)$ under $Gal(H_{\mathcal{O}}/K)$.

2. Preliminary

Let $\mathcal{Q}_D^0$ be the set of primitive quadratic forms and $C(D) = \mathcal{Q}_D^0/\Gamma(1)$ denote the form class group of discriminant D. Since $Gal(H_{\mathcal{O}}/K)$ is isomorphic to $C(D)$, it suffices to compute the action of a primitive quadratic form $Q = [a, b, c]$ on the class invariant $\frac{1}{\sqrt{-3}}\mathfrak{g}_2^2(\theta)$.

THEOREM 2.1. [2, 3] *Let $\mathbb{Z}[\theta]$ be the ring of integers of an imaginary quadratic number field K of discriminant D and let $Q = [a, b, c]$ be a primitive quadratic form of discriminant D. Let $\theta = \frac{-B+\sqrt{D}}{2}$ as defined in (1.1) and $\tau_Q = \frac{-b+\sqrt{-D}}{2a}$. Let $M = M_{[a,b,c]} \in \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$ be given as follows: For $D \equiv 0 \pmod 4$,*

$$M \equiv \begin{cases} \begin{pmatrix} a & \frac{b}{2} \\ 0 & 1 \end{pmatrix} & \pmod{p^{r_p}} \quad \text{if } p \nmid a; \\ \begin{pmatrix} -\frac{b}{2} & -c \\ 1 & 0 \end{pmatrix} & \pmod{p^{r_p}} \quad \text{if } p \mid a \text{ and } p \nmid c; \\ \begin{pmatrix} -\frac{b}{2} - a & -\frac{b}{2} - c \\ 1 & -1 \end{pmatrix} & \pmod{p^{r_p}} \quad \text{if } p \mid a \text{ and } p \mid c, \end{cases} \tag{2.1}$$

and for $D \equiv 1 \pmod 4$,

$$
(2.2)\quad M \equiv \begin{cases} \begin{pmatrix} a & \frac{b-1}{2} \\ 0 & 1 \end{pmatrix} & \pmod{p^{r_p}} \ \text{ if } p \nmid a; \\ \begin{pmatrix} \frac{-b-1}{2} & -c \\ 1 & 0 \end{pmatrix} & \pmod{p^{r_p}} \ \text{ if } p \mid a \text{ and } p \nmid c; \\ \begin{pmatrix} \frac{-b-1}{2} - a & -\frac{1-b}{2} - c \\ 1 & -1 \end{pmatrix} & \pmod{p^{r_p}} \ \text{ if } p \mid a \text{ and } p \mid c. \end{cases}
$$

where p *runs over all prime factors of* N *and* $p^{r_p}||N$. *Then the Galois action of the class of* $[a, -b, c]$ *in* $C(D)$ *with respect to the Artin map is given by*

$$f(\theta)^{[a,-b,c]} = f^M(\tau_Q)$$

for any modular function f *of level* N *such that* $f(\theta) \in H_{\mathcal{O}}$. *Here* f^M *denote the image of* f *under the action of* M.

The action of M depends only on M_m for all primes $p|N$ where $M_m \in \mathrm{GL}_2(\mathbb{Z}/m\mathbb{Z})$ is the reduction modulo m of M. Every M_m with determinant x decomposes as $M_m = \begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ for some $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}/m\mathbb{Z})$. Since $\mathrm{SL}_2(\mathbb{Z}/m\mathbb{Z})$ is generated by $S_m \equiv \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $T_m \equiv \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, it suffices to find the action of $\begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix}_{p^{r_p}}$, $S_{p^{r_p}}$ and $T_{p^{r_p}}$ on f for all $p|N$. Denote ζ_n by a primitive nth root of unity. For $\begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix}_{p^{r_p}}$, the action on f is given by lifting the automorphism of $\mathbb{Q}(\zeta_N)$ determined by

$$\zeta_{p^{r_p}} \mapsto \zeta_{p^{r_p}}^x \quad \text{and} \quad \zeta_{q^{r_q}} \mapsto \zeta_{q^{r_q}}$$

for all prime factors $q|N$ with $q \neq p$. In order that the actions of the matrices at different primes commute with each other, we lift $S_{p^{r_p}}$ and $T_{p^{r_p}}$ to matrices in $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ such that they reduce to the identity matrix in $\mathrm{SL}_2(\mathbb{Z}/q^{r_q}\mathbb{Z})$ for all $q \neq p$.

The Dedekind-eta function

$$\eta(z) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n), \quad \text{with} \quad q = e^{2\pi i z}$$

is holomorphic and non-zero for z in the complex upper half plane $\mathbb{H}$ and $\Delta(z) = \eta^{24}(z)$ is modular form of weight 12 with no poles or zeros on $\mathbb{H}$. Then we have generalized Weber functions as follows:

(2.3)

$$\mathfrak{g}_0(z) = \frac{\eta(\frac{z}{3})}{\eta(z)},\ \mathfrak{g}_1(z) = \zeta_{24}^{-1}\frac{\eta(\frac{z+1}{3})}{\eta(z)},\ \mathfrak{g}_2(z) = \frac{\eta(\frac{z+2}{3})}{\eta(z)},\ \mathfrak{g}_3(z) = \sqrt{3}\frac{\eta(3z)}{\eta(z)}.$$

Note that the functions in (2.3) are modular of level 72. For the generating matrices $S,\ T \in SL_2(\mathbb{Z})$ given by $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, the transformation rules $\eta \circ S(z) = \sqrt{-iz}\eta(z)$ and $\eta \circ T(z) = \zeta_{24}\eta(z)$ hold. Hence

$$\begin{aligned} (\mathfrak{g}_0, \mathfrak{g}_1, \mathfrak{g}_2, \mathfrak{g}_3) \circ S &= (\mathfrak{g}_3, \zeta_{24}^{-2}\mathfrak{g}_2, \zeta_{24}^{2}\mathfrak{g}_1, \mathfrak{g}_0), \\ (\mathfrak{g}_0, \mathfrak{g}_1, \mathfrak{g}_2, \mathfrak{g}_3) \circ T &= (\mathfrak{g}_1, \zeta_{24}^{-2}\mathfrak{g}_2, \mathfrak{g}_0, \zeta_{24}^{2}\mathfrak{g}_3). \end{aligned} \tag{2.4}$$

3. Results

In this section, we compute the action of a primitive quadratic form $Q = [a, b, c]$ on the class invariant $\frac{1}{\sqrt{-3}}\mathfrak{g}_2^2(\theta)$. For that we need to find the action of $M_m \in GL_2(\mathbb{Z}/m\mathbb{Z})$ with $m = 8, 9$. Combining Lemma 6 of [3] and the transformation rule (2.4), we obtain the following:

Lemma 3.1. *The actions of* $\begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix}_m$, S_m *and* T_m $(m = 8, 9)$ *on* $\mathfrak{g}_i^2$ $(i = 0, 1, 2, 3)$ *are given by*

	$\mathfrak{g}_0^2$	$\mathfrak{g}_1^2$	$\mathfrak{g}_2^2$	$\mathfrak{g}_3^2$
$\begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix}_8$	$\mathfrak{g}_0^2$	$\mathfrak{g}_1^2$	$\mathfrak{g}_2^2$	$\mathfrak{g}_3^2$
S_8	$-\mathfrak{g}_0^2$	$-\mathfrak{g}_1^2$	$-\mathfrak{g}_2^2$	$-\mathfrak{g}_3^2$
T_8	$-\mathfrak{g}_0^2$	$-\mathfrak{g}_1^2$	$-\mathfrak{g}_2^2$	$-\mathfrak{g}_3^2$
$\begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix}_9$, $x = -3k+1$	$\mathfrak{g}_0^2$	$\zeta_3^{2k}\mathfrak{g}_1^2$	$\zeta_3^{k}\mathfrak{g}_2^2$	$\mathfrak{g}_3^2$
$\begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix}_9$, $x = -3k-1$	$\mathfrak{g}_0^2$	$\zeta_3^{2k}\mathfrak{g}_2^2$	$\zeta_3^{k}\mathfrak{g}_1^2$	$\mathfrak{g}_3^2$
S_9	$-\mathfrak{g}_3^2$	$\zeta_3\mathfrak{g}_2^2$	$\zeta_3^2\mathfrak{g}_1^2$	$-\mathfrak{g}_0^2$
T_9	$-\mathfrak{g}_1^2$	$\zeta_3\mathfrak{g}_2^2$	$-\mathfrak{g}_0^2$	$\zeta_3^2\mathfrak{g}_3^2$

Using this, together with Theorem 2.1, we have the following theorems.

THEOREM 3.2. *Let $D \equiv -3 \pmod{36}$ be a discriminant of an order $\mathcal{O} = [\theta, 1]$ in an imaginary quadratic field. Let $\theta = \frac{-1+\sqrt{D}}{2}$, $\tau_Q = \frac{-b+\sqrt{D}}{2a}$ and $u = (-1)^{\frac{b+1}{2}+ac+a+c}$. If $[a, b, c]$ be a reduced primitive quadratic form of discriminant D, then the actions of $[a, -b, c]$ on $\frac{1}{\sqrt{-3}}\mathfrak{g}_2^2(\theta)$ are as follows:*

(1) *The case $3 \nmid a$.*

a) *If $b \equiv 0 \pmod 3$, then $\frac{1}{\sqrt{-3}}\mathfrak{g}_2^2(\theta)^{[a,-b,c]}$ is given by the following table:*

	$b \equiv 0 \pmod 9$	$b \equiv 3 \pmod 9$	$b \equiv 6 \pmod 9$
$a \equiv 1 \pmod 9$	$-\frac{u\zeta_3}{\sqrt{-3}}\mathfrak{g}_0^2(\tau_Q)$	$-\frac{u}{\sqrt{-3}}\mathfrak{g}_0^2(\tau_Q)$	$\frac{u\zeta_3^2}{\sqrt{-3}}\mathfrak{g}_0^2(\tau_Q)$
$a \equiv 2 \pmod 9$	$\frac{u\zeta_3^2}{\sqrt{-3}}\mathfrak{g}_0^2(\tau_Q)$	$\frac{u}{\sqrt{-3}}\mathfrak{g}_0^2(\tau_Q)$	$\frac{u\zeta_3}{\sqrt{-3}}\mathfrak{g}_0^2(\tau_Q)$
$a \equiv 4 \pmod 9$	$-\frac{u\zeta_3}{\sqrt{-3}}\mathfrak{g}_0^2(\tau_Q)$	$-\frac{u}{\sqrt{-3}}\mathfrak{g}_0^2(\tau_Q)$	$-\frac{u\zeta_3^2}{\sqrt{-3}}\mathfrak{g}_0^2(\tau_Q)$
$a \equiv 5 \pmod 9$	$\frac{u\zeta_3^2}{\sqrt{-3}}\mathfrak{g}_0^2(\tau_Q)$	$\frac{u}{\sqrt{-3}}\mathfrak{g}_0^2(\tau_Q)$	$\frac{u\zeta_3}{\sqrt{-3}}\mathfrak{g}_0^2(\tau_Q)$
$a \equiv 7 \pmod 9$	$\frac{-u\zeta_3}{\sqrt{-3}}\mathfrak{g}_0^2(\tau_Q)$	$-\frac{u}{\sqrt{-3}}\mathfrak{g}_0^2(\tau_Q)$	$-\frac{u\zeta_3^2}{\sqrt{-3}}\mathfrak{g}_0^2(\tau_Q)$
$a \equiv 8 \pmod 9$	$\frac{u\zeta_3^2}{\sqrt{-3}}\mathfrak{g}_0^2(\tau_Q)$	$\frac{u}{\sqrt{-3}}\mathfrak{g}_0^2(\tau_Q)$	$\frac{u\zeta_3}{\sqrt{-3}}\mathfrak{g}_0^2(\tau_Q)$

b) *If $a + b \equiv 0 \pmod 3$, then $\frac{1}{\sqrt{-3}}\mathfrak{g}_2^2(\theta)^{[a,-b,c]}$ is given by the following table:*

	$b \equiv 2 \pmod 9$	$b \equiv 5 \pmod 9$	$b \equiv 8 \pmod 9$
$a \equiv 1 \pmod 9$	$\frac{u\zeta_3}{\sqrt{-3}}\mathfrak{g}_1^2(\tau_Q)$	$\frac{u}{\sqrt{-3}}\mathfrak{g}_1^2(\tau_Q)$	$\frac{u\zeta_3^2}{\sqrt{-3}}\mathfrak{g}_1^2(\tau_Q)$
$a \equiv 4 \pmod 9$	$\frac{u}{\sqrt{-3}}\mathfrak{g}_1^2(\tau_Q)$	$\frac{u\zeta_3^2}{\sqrt{-3}}\mathfrak{g}_1^2(\tau_Q)$	$\frac{u\zeta_3}{\sqrt{-3}}\mathfrak{g}_1^2(\tau_Q)$
$a \equiv 7 \pmod 9$	$\frac{u\zeta_3^2}{\sqrt{-3}}\mathfrak{g}_1^2(\tau_Q)$	$\frac{u\zeta_3}{\sqrt{-3}}\mathfrak{g}_1^2(\tau_Q)$	$\frac{u}{\sqrt{-3}}\mathfrak{g}_1^2(\tau_Q)$

	$b \equiv 1 \pmod 9$	$b \equiv 4 \pmod 9$	$b \equiv 7 \pmod 9$
$a \equiv 2 \pmod 9$	$-\frac{u\zeta_3}{\sqrt{-3}}\mathfrak{g}_1^2(\tau_Q)$	$-\frac{u\zeta_3^2}{\sqrt{-3}}\mathfrak{g}_1^2(\tau_Q)$	$-\frac{u}{\sqrt{-3}}\mathfrak{g}_1^2(\tau_Q)$
$a \equiv 5 \pmod 9$	$-\frac{u\zeta_3^2}{\sqrt{-3}}\mathfrak{g}_1^2(\tau_Q)$	$-\frac{u}{\sqrt{-3}}\mathfrak{g}_1^2(\tau_Q)$	$-\frac{u\zeta_3}{\sqrt{-3}}\mathfrak{g}_1^2(\tau_Q)$
$a \equiv 8 \pmod 9$	$-\frac{u}{\sqrt{-3}}\mathfrak{g}_1^2(\tau_Q)$	$-\frac{u\zeta_3}{\sqrt{-3}}\mathfrak{g}_1^2(\tau_Q)$	$-\frac{u\zeta_3^2}{\sqrt{-3}}\mathfrak{g}_1^2(\tau_Q)$

c) *If* $b \not\equiv 0 \pmod 3$ *and* $a+b \equiv \pm 1 \pmod 3$, *then* $\frac{1}{\sqrt{-3}}\mathfrak{g}_2^2(\theta)^{[a,-b,c]}$ *is given by the following table:*

	$b \equiv 1 \pmod 9$	$b \equiv 4 \pmod 9$	$b \equiv 7 \pmod 9$
$a \equiv 1 \pmod 9$	$\frac{u}{\sqrt{-3}}\mathfrak{g}_2^2(\tau_Q)$	$\frac{u\zeta_3^2}{\sqrt{-3}}\mathfrak{g}_2^2(\tau_Q)$	$\frac{u\zeta_3}{\sqrt{-3}}\mathfrak{g}_2^2(\tau_Q)$
$a \equiv 4 \pmod 9$	$\frac{u\zeta_3}{\sqrt{-3}}\mathfrak{g}_2^2(\tau_Q)$	$\frac{u}{\sqrt{-3}}\mathfrak{g}_2^2(\tau_Q)$	$\frac{u\zeta_3^2}{\sqrt{-3}}\mathfrak{g}_2^2(\tau_Q)$
$a \equiv 7 \pmod 9$	$\frac{u\zeta_3^2}{\sqrt{-3}}\mathfrak{g}_2^2(\tau_Q)$	$\frac{u\zeta_3}{\sqrt{-3}}\mathfrak{g}_2^2(\tau_Q)$	$\frac{u}{\sqrt{-3}}\mathfrak{g}_2^2(\tau_Q)$

	$b \equiv 2 \pmod 9$	$b \equiv 5 \pmod 9$	$b \equiv 8 \pmod 9$
$a \equiv 2 \pmod 9$	$-\frac{u\zeta_3}{\sqrt{-3}}\mathfrak{g}_2^2(\tau_Q)$	$-\frac{u\zeta_3^2}{\sqrt{-3}}\mathfrak{g}_2^2(\tau_Q)$	$-\frac{u}{\sqrt{-3}}\mathfrak{g}_2^2(\tau_Q)$
$a \equiv 5 \pmod 9$	$-\frac{u}{\sqrt{-3}}\mathfrak{g}_2^2(\tau_Q)$	$-\frac{u\zeta_3}{\sqrt{-3}}\mathfrak{g}_2^2(\tau_Q)$	$-\frac{u\zeta_3^2}{\sqrt{-3}}\mathfrak{g}_2^2(\tau_Q)$
$a \equiv 8 \pmod 9$	$-\frac{u\zeta_3^2}{\sqrt{-3}}\mathfrak{g}_2^2(\tau_Q)$	$-\frac{u}{\sqrt{-3}}\mathfrak{g}_2^2(\tau_Q)$	$-\frac{u\zeta_3}{\sqrt{-3}}\mathfrak{g}_2^2(\tau_Q)$

(2) *The cases* $3|a$ *and* $3 \nmid c$.

a) *If* $b \equiv 0 \pmod 3$, *then* $\frac{1}{\sqrt{-3}}\mathfrak{g}_2^2(\theta)^{[a,-b,c]}$ *is given by the following table:*

	$b \equiv 0 \pmod 9$	$b \equiv 3 \pmod 9$	$b \equiv 6 \pmod 9$
$c \equiv 1 \pmod 9$	$\frac{u\zeta_3}{\sqrt{-3}}\mathfrak{g}_3^2(\tau_Q)$	$\frac{u\zeta_3^2}{\sqrt{-3}}\mathfrak{g}_3^2(\tau_Q)$	$\frac{u}{\sqrt{-3}}\mathfrak{g}_3^2(\tau_Q)$
$c \equiv 2 \pmod 9$	$-\frac{u\zeta_3^2}{\sqrt{-3}}\mathfrak{g}_3^2(\tau_Q)$	$-\frac{u\zeta_3}{\sqrt{-3}}\mathfrak{g}_3^2(\tau_Q)$	$-\frac{u}{\sqrt{-3}}\mathfrak{g}_3^2(\tau_Q)$
$c \equiv 4 \pmod 9$	$\frac{u\zeta_3}{\sqrt{-3}}\mathfrak{g}_3^2(\tau_Q)$	$\frac{u\zeta_3^2}{\sqrt{-3}}\mathfrak{g}_3^2(\tau_Q)$	$\frac{u}{\sqrt{-3}}\mathfrak{g}_3^2(\tau_Q)$
$c \equiv 5 \pmod 9$	$-\frac{u\zeta_3^2}{\sqrt{-3}}\mathfrak{g}_3^2(\tau_Q)$	$-\frac{u\zeta_3}{\sqrt{-3}}\mathfrak{g}_3^2(\tau_Q)$	$-\frac{u}{\sqrt{-3}}\mathfrak{g}_3^2(\tau_Q)$
$c \equiv 7 \pmod 9$	$\frac{u\zeta_3}{\sqrt{-3}}\mathfrak{g}_3^2(\tau_Q)$	$\frac{u\zeta_3^2}{\sqrt{-3}}\mathfrak{g}_3^2(\tau_Q)$	$\frac{u}{\sqrt{-3}}\mathfrak{g}_3^2(\tau_Q)$
$c \equiv 8 \pmod 9$	$-\frac{u\zeta_3^2}{\sqrt{-3}}\mathfrak{g}_3^2(\tau_Q)$	$-\frac{u\zeta_3}{\sqrt{-3}}\mathfrak{g}_3^2(\tau_Q)$	$-\frac{u}{\sqrt{-3}}\mathfrak{g}_3^2(\tau_Q)$

b) *If* $b \not\equiv 0 \pmod 3$ *and* $b+c \equiv \pm 1 \pmod 3$, *then* $\frac{1}{\sqrt{-3}}\mathfrak{g}_2^2(\theta)^{[a,-b,c]}$ *is given by the following table:*

	$b \equiv 1 \pmod 9$	$b \equiv 4 \pmod 9$	$b \equiv 7 \pmod 9$
$a \equiv 1 \pmod 9$	$\frac{u}{\sqrt{-3}}\mathfrak{g}_2^2(\tau_Q)$	$\frac{u\zeta_3}{\sqrt{-3}}\mathfrak{g}_2^2(\tau_Q)$	$\frac{u\zeta_3^2}{\sqrt{-3}}\mathfrak{g}_2^2(\tau_Q)$
$a \equiv 4 \pmod 9$	$\frac{u\zeta_3^2}{\sqrt{-3}}\mathfrak{g}_2^2(\tau_Q)$	$\frac{u}{\sqrt{-3}}\mathfrak{g}_2^2(\tau_Q)$	$\frac{u\zeta_3}{\sqrt{-3}}\mathfrak{g}_2^2(\tau_Q)$

$a \equiv 7 \pmod 9$	$\frac{u\zeta_3}{\sqrt{-3}}\mathfrak{g}_2^2(\tau_Q)$	$\frac{u\zeta_3^2}{\sqrt{-3}}\mathfrak{g}_2^2(\tau_Q)$	$\frac{u}{\sqrt{-3}}\mathfrak{g}_2^2(\tau_Q)$

	$b \equiv 2 \pmod 9$	$b \equiv 5 \pmod 9$	$b \equiv 8 \pmod 9$
$a \equiv 2 \pmod 9$	$-\frac{u\zeta_3}{\sqrt{-3}}\mathfrak{g}_2^2(\tau_Q)$	$-\frac{u}{\sqrt{-3}}\mathfrak{g}_2^2(\tau_Q)$	$-\frac{u\zeta_3^2}{\sqrt{-3}}\mathfrak{g}_2^2(\tau_Q)$
$a \equiv 5 \pmod 9$	$-\frac{u\zeta_3^2}{\sqrt{-3}}\mathfrak{g}_2^2(\tau_Q)$	$-\frac{u\zeta_3}{\sqrt{-3}}\mathfrak{g}_2^2(\tau_Q)$	$-\frac{u}{\sqrt{-3}}\mathfrak{g}_2^2(\tau_Q)$
$a \equiv 8 \pmod 9$	$-\frac{u}{\sqrt{-3}}\mathfrak{g}_2^2(\tau_Q)$	$-\frac{u\zeta_3^2}{\sqrt{-3}}\mathfrak{g}_2^2(\tau_Q)$	$-\frac{u\zeta_3}{\sqrt{-3}}\mathfrak{g}_2^2(\tau_Q)$

c) *If* $b + c \equiv 0 \pmod 3$, *then* $\frac{1}{\sqrt{-3}}\mathfrak{g}_2^2(\theta)^{[a,-b,c]}$ *is given by the following table:*

	$b \equiv 2 \pmod 9$	$b \equiv 5 \pmod 9$	$b \equiv 8 \pmod 9$
$a \equiv 1 \pmod 9$	$\frac{u}{\sqrt{-3}}\mathfrak{g}_1^2(\tau_Q)$	$\frac{u\zeta_3}{\sqrt{-3}}\mathfrak{g}_1^2(\tau_Q)$	$\frac{u\zeta_3^2}{\sqrt{-3}}\mathfrak{g}_1^2(\tau_Q)$
$a \equiv 4 \pmod 9$	$\frac{u\zeta_3}{\sqrt{-3}}\mathfrak{g}_1^2(\tau_Q)$	$\frac{u\zeta_3^2}{\sqrt{-3}}\mathfrak{g}_1^2(\tau_Q)$	$\frac{u}{\sqrt{-3}}\mathfrak{g}_1^2(\tau_Q)$
$a \equiv 7 \pmod 9$	$\frac{u\zeta_3^2}{\sqrt{-3}}\mathfrak{g}_1^2(\tau_Q)$	$\frac{u}{\sqrt{-3}}\mathfrak{g}_1^2(\tau_Q)$	$\frac{u\zeta_3}{\sqrt{-3}}\mathfrak{g}_1^2(\tau_Q)$

	$b \equiv 1 \pmod 9$	$b \equiv 4 \pmod 9$	$b \equiv 7 \pmod 9$
$a \equiv 2 \pmod 9$	$-\frac{u\zeta_3^2}{\sqrt{-3}}\mathfrak{g}_1^2(\tau_Q)$	$-\frac{u\zeta_3}{\sqrt{-3}}\mathfrak{g}_1^2(\tau_Q)$	$-\frac{u}{\sqrt{-3}}\mathfrak{g}_1^2(\tau_Q)$
$a \equiv 5 \pmod 9$	$-\frac{u\zeta_3}{\sqrt{-3}}\mathfrak{g}_1^2(\tau_Q)$	$-\frac{u}{\sqrt{-3}}\mathfrak{g}_1^2(\tau_Q)$	$-\frac{u\zeta_3^2}{\sqrt{-3}}\mathfrak{g}_1^2(\tau_Q)$
$a \equiv 8 \pmod 9$	$-\frac{u}{\sqrt{-3}}\mathfrak{g}_1^2(\tau_Q)$	$-\frac{u\zeta_3^2}{\sqrt{-3}}\mathfrak{g}_1^2(\tau_Q)$	$-\frac{u\zeta_3}{\sqrt{-3}}\mathfrak{g}_1^2(\tau_Q)$

(3) *The cases* $3|a$ *and* $3|c$.

a) *If* $a - c \equiv -3 \pmod 9$, *then*

$$\frac{1}{\sqrt{-3}}\mathfrak{g}_2^2(\theta)^{[a,-b,c]} = \begin{cases} \frac{u\zeta_3}{\sqrt{-3}}\mathfrak{g}_1^2(\tau_Q) & \text{if } b \equiv 1 \pmod 3; \\ -\frac{u\zeta_3}{\sqrt{-3}}\mathfrak{g}_1^2(\tau_Q) & \text{if } b \equiv 2 \pmod 3; \end{cases}$$

b) *If* $a - c \equiv 0 \pmod 9$, *then*

$$\frac{1}{\sqrt{-3}}\mathfrak{g}_2^2(\theta)^{[a,-b,c]} = \begin{cases} \frac{u\zeta_3^2}{\sqrt{-3}}\mathfrak{g}_1^2(\tau_Q) & \text{if } b \equiv 1 \pmod 3; \\ -\frac{u}{\sqrt{-3}}\mathfrak{g}_1^2(\tau_Q) & \text{if } b \equiv 2 \pmod 3; \end{cases}$$

c) *If* $a - c \equiv 3 \pmod 9$, *then*

$$\frac{1}{\sqrt{-3}}\mathfrak{g}_2^2(\theta)^{[a,-b,c]} = \begin{cases} \frac{u}{\sqrt{-3}}\mathfrak{g}_1^2(\tau_Q) & \text{if } b \equiv 1 \pmod 3; \\ -\frac{u\zeta_3^2}{\sqrt{-3}}\mathfrak{g}_1^2(\tau_Q) & \text{if } b \equiv 2 \pmod 3; \end{cases}$$

Acknowledgments

We thank S.-Y. Kang and C.H. Kim for their kind and valuable suggestions in proving the results of this paper.

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