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STRONG CONVERGENCE OF HYBRID PROJECTION METHODS FOR QUASI–ϕ–NONEXPANSIVE MAPPINGS

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Abstract. In this paper, we consider the convergence of the shrinking projection method for quasi-ϕ-nonexpansive mappings. Strong convergence theorems are established in a uniformly smooth and strictly convex Banach space which enjoys the Kadec-Klee property.

1. Introduction and preliminaries

Let E be a real Banach space and let C be a nonempty subset of E. Let $T : C \to C$ be a nonlinear mapping. A point $x \in C$ is a *fixed point* of T provided $Tx = x$. In this paper, we use $F(T)$ to denote the fixed point set of T and use $\to$ and $\rightharpoonup$ to denote the strong convergence and weak convergence, respectively.

Recall that T is said to be *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$.

Recall that the normal Mann iterative process was introduced by Mann [8] in 1953. Since then, construction of fixed points for nonexpansive mappings via the normal Mann's iterative process has been extensively investigated by many authors. The normal Mann's iterative process generates a sequence $\{x_n\}$ in the following manner:

$$\forall x_1 \in C, \quad x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T x_n, \quad \forall n \geq 1, \tag{1.1}$$

where the sequence $\{\alpha_n\}$ is in the interval $(0, 1)$.

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If T is a nonexpansive mapping with a fixed point and the control sequence $\{\alpha_n\}$ is chosen so that $\sum_{n=0}^{\infty} \alpha_n(1-\alpha_n) = \infty$, then the sequence $\{x_n\}$ generated by normal Mann's iterative process (1.1) converges weakly to a fixed point of T (this is also valid in a uniformly convex Banach space with the Fréchet differentiable norm [13]). It is well known that, in an infinite-dimensional Hilbert space, the normal Mann's iterative algorithm has only weak convergence, in general, even for nonexpansive mappings.

Attempts to modify the normal Mann iteration (1.1) for nonexpansive mappings by hybrid projection algorithms so that strong convergence is guaranteed have recently been made. Nakajo and Takahashi [10] proposed the following modification of the Mann iteration for a single nonexpansive mapping T in a Hilbert space. To be more precise, they proved the following theorem.

Theorem 1.1. *Let C be a closed convex subset of a Hilbert space H and let $T : C \to C$ be a nonexpansive mapping such that $F(T) \neq \emptyset$. Assume that $\{\alpha_n\}$ is a sequence in $[0, 1]$ such that $\alpha_n \leq 1 - \delta$ for some $\delta \in (0, 1]$. Define a sequence $\{x_n\}$ in C by the following algorithm:*

$$(1.2) \qquad \begin{cases} x_0 \in C \quad \text{chosen arbitrarily}, \\ y_n = \alpha_n x_n + (1-\alpha_n) T x_n, \\ C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_0 - x_n, x_n - z \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \quad \forall n \geq 0. \end{cases}$$

Then $\{x_n\}$ converges in norm to $P_{F(T)} x_0$.

In 2008, Takahashi, Takeuchi and Kubota [15] introduced so called shrinking projection methods in a Hilbert space for nonexpansive mappings. To be more precise, they obtain the following theorem.

Theorem 1.2. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let T be a nonexpansive mapping of C into itself such that $F(T) \neq \emptyset$ and let $x_0 \in H$. For $C_1 = C$ and $u_1 = P_{C_1} x_0$, define a* *sequence* $\{u_n\}$ *of* C *as follows:*

$$(1.3)\qquad \begin{cases} y_n = \alpha_n u_n + (1-\alpha_n)Tu_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \le \|u_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}}x_0, \quad \forall n \ge 0, \end{cases}$$

where $0 \le \alpha_n \le a < 1$ *for all* $n \ge 0$. *Then* $\{u_n\}$ *converges strongly to* $z_0 = P_{F(T)}x_0$.

Recently, many authors further considered the problem of modifying normal Mann iterative process in the framework of real Banach spaces. Before proceeding further, we give some definitions and propositions in Banach spaces first.

Let E be a Banach space with dual E^*. We denote by the *normalized duality mapping* J from E to 2^{E^*} defined by

$$Jx = \{f^* \in E^* : \langle x, f^*\rangle = \|x\|^2 = \|f^*\|^2\},$$

where $\langle\cdot,\cdot\rangle$ denotes the generalized duality pairing.

A Banach space E is said to be *strictly convex* if $\left\|\frac{x+y}{2}\right\| < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \ne y$. It is said to be *uniformly convex* if $\lim_{n\to\infty}\|x_n - y_n\| = 0$ for any two sequences $\{x_n\}$ and $\{y_n\}$ in E such that $\|x_n\| = \|y_n\| = 1$ and $\lim_{n\to\infty}\left\|\frac{x_n+y_n}{2}\right\| = 1$. Let $U = \{x \in E : \|x\| = 1\}$ be the unit sphere of E. Then the Banach space E is said to be *smooth* provided $\lim_{t\to 0}\frac{\|x+ty\|-\|x\|}{t}$ exists for each $x, y \in U$. It is also said to be *uniformly smooth* if the limit is attained uniformly for $x, y \in E$. It is well known that if E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E. It is also well known that if E is uniformly smooth if and only if E^* is uniformly convex.

Recall that a Banach space E has the Kadec-Klee property if for any sequence $\{x_n\} \subset E$ and $x \in E$ with $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$, then $\|x_n - x\| \to 0$ as $n \to \infty$. For more details on Kadec-Klee property, the readers is referred to [7] and the references therein. It is well known that if E is a uniformly convex Banach spaces, then E enjoys the Kadec-Klee property.

As we all know that if C is a nonempty closed convex subset of a Hilbert space H and $P_C : H \to C$ is the metric projection of H onto C, then P_C is nonexpansive. This fact actually characterizes Hilbert spaces and consequently, it is not available in more general Banach spaces. In this connection, Alber [1] recently introduced a generalized projection operator Π_C in a Banach space E which is an analogue of the metric projection in Hilbert spaces.

Next, we assume that E is a smooth Banach space. Consider the functional defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2, \quad \forall x, y \in E. \tag{1.4}$$

Observe that, in a Hilbert space H, (1.4) is reduced to $\phi(x, y) = \|x-y\|^2$ for all $x, y \in H$. The generalized projection $\Pi_C : E \to C$ is a map that assigns to an arbitrary point $x \in E$ the minimum point of the functional $\phi(x, y)$, that is, $\Pi_C x = \bar{x}$, where $\bar{x}$ is the solution to the minimization problem

$$\phi(\bar{x}, x) = \inf_{y \in C} \phi(y, x) \tag{1.5}$$

existence and uniqueness of the operator Π_C follows from the properties of the functional $\phi(x, y)$ and strict monotonicity of the mapping J (see, for example, [1], [2], [6] and [14]). In Hilbert spaces, $\Pi_C = P_C$. It is obvious from the definition of function ϕ that

$$(\|y\| - \|x\|)^2 \le \phi(y, x) \le (\|y\| + \|x\|)^2, \quad \forall x, y \in E. \tag{1.6}$$

REMARK 1.1. If E is a reflexive, strictly convex and smooth Banach space, then for $x, y \in E$, $\phi(x, y) = 0$ if and only if $x = y$. It is sufficient to show that if $\phi(x, y) = 0$ then $x = y$. From (1.6), we have $\|x\| = \|y\|$. This implies $\langle x, Jy\rangle = \|x\|^2 = \|Jy\|^2$. From the definition of J, one has $Jx = Jy$. Therefore, we have $x = y$, see [6] and [14] for more details.

Let C be a nonempty closed convex subset of E and let T be a mapping from C into itself. A point p in C is said to be an *asymptotic fixed point* of T [13] if C contains a sequence $\{x_n\}$ which converges weakly to p such that

$\lim_{n\to\infty}\|x_n - Tx_n\| = 0$. The set of asymptotic fixed points of T will be denoted by $\widetilde{F}(T)$. A mapping T from C into itself is said to be *relatively nonexpansive* [3]-[5] if $\widetilde{F}(T) = F(T) \neq \emptyset$ and $\phi(p,Tx) \leq \phi(p,x)$ for all $x \in C$ and $p \in F(T)$. The asymptotic behavior of a relatively nonexpansive mappings was studied in [3]-[5].

The mapping T is said to be *ϕ-nonexpansive* if $\phi(Tx,Ty) \leq \phi(x,y)$ for all $x, y \in C$. T is said to be *quasi-ϕ-nonexpansive* if $F(T) \neq \emptyset$ and $\phi(p,Tx) \leq \phi(p,x)$ for all $x \in C$ and $p \in F(T)$ (see [11], [12], [16] and [17]).

REMARK 1.2. The class of quasi-ϕ-nonexpansive mappings is more general than the class of relatively nonexpansive mappings which requires the strong restriction: $F(T) = \widetilde{F}(T)$.

Recently, Matsushita and Takahashi [9] improved Theorem 1.1 from Hilbert spaces to Banach spaces. To be more precise, they proved the following theorem.

THEOREM 1.3. *Let E be a uniformly convex and uniformly smooth Banach space and let C be a nonempty closed convex subset of E. Let T be a relatively nonexpansive mapping from C into itself and let $\{\alpha_n\}$ be a sequence of real numbers such that $0 \leq \alpha_n < 1$ and $\limsup_{n\to\infty}\alpha_n < 1$. Suppose that $\{x_n\}$ is given by*

$$\begin{cases} x_0 = x \in C, \\ y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JTx_n), \\ H_n = \{z \in C : \phi(z,y_n) \leq \phi(z,x_n)\}, \\ W_n = \{z \in C : \langle x_n - z, Jx - Jx_n\rangle \geq 0\}, \\ x_{n+1} = P_{H_n\cap W_n}x_0, \quad \forall n \geq 0, \end{cases} \tag{1.7}$$

where J is the duality mapping on E. If $F(T)$ is nonempty, then $\{x_n\}$ converges strongly to $P_{F(T)}x$, where $P_{F(T)}$ is the generalized projection from C onto $F(T)$.

In this paper, motivated by Theorems 1.1∼1.3, we re-consider the problem of modifying normal Mann iteration to obtain strong convergence based on shrinking projection methods. Strong convergence theorems are established in the framework of real Banach spaces. The results presented in this paper improves the corresponding results in Matsushita and Takahashi [9], Nakajo and Takahashi [10] and Takahashi, Takeuchi and Kubota [15].

We need the following lemmas for the proof of our main results.

LEMMA 1.1. ([1]) *Let C be a nonempty closed convex subset of a smooth Banach space E and $x \in E$. Then, $x_0 = \Pi_C x$ if and only if*

$$\langle x_0 - y, Jx - Jx_0 \rangle \geq 0, \quad \forall y \in C.$$

LEMMA 1.2. ([1]) *Let E be a reflexive, strictly convex and smooth Banach space, let C be a nonempty closed convex subset of E and $x \in E$. Then*

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \leq \phi(y, x), \quad \forall y \in C.$$

The following lemma can be deduced from Zhou and Gao [16].

LEMMA 1.3. *Let E be a reflexive, strictly convex and smooth Banach space and let C be a nonempty closed convex subset of E. Let $T : C \to C$ be a quasi-ϕ-nonexpansive mapping. Then $F(T)$ is a closed convex subset of C.*

2. Main results

Now, we are ready to give our main results.

THEOREM 2.1. *Let E be a uniformly smooth and strictly convex Banach space which enjoys the Kadec-Klee property and let C be a nonempty closed convex subset of E. Let $T : C \to C$ be a closed quasi-ϕ-nonexpansive mapping. Let $\{x_n\}$ be a sequence generated in the following manner:*

$$\begin{cases} x_0 \in E \quad \text{chosen arbitrarily}, \\ C_1 = C, \\ x_1 = \Pi_{C_1} x_0, \\ y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n) JTx_n), \\ C_{n+1} = \{z \in C_n : \phi(z, y_n) \leq \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \quad \forall n \geq 0. \end{cases} \tag{2.1}$$

If the control sequence $\{\alpha_n\}$ *satisfies the restrictions:* $0 \leq \alpha_n < 1$ *and* $\limsup_{n\to\infty} \alpha_n < 1$, *then* $\{x_n\}$ *converges strongly to* $\Pi_{F(T)}x_0$.

Proof. First, we show that C_n is closed and convex for all $n \geq 1$. It is obvious that $C_1 = C$ is closed and convex. Suppose that C_k is closed and convex for some k. For $z \in C_k$, we see that $\phi(z, y_k) \leq \phi(z, x_k)$ is equivalent to

$$2\langle z, Jx_k - Jy_k\rangle \leq \|x_k\|^2 - \|y_k\|^2.$$

It is to see that C_{k+1} is closed and convex. Then, for all $n \geq 1$, C_n is closed and convex. This shows that $\Pi_{C_{n+1}}x_0$ is well defined. Next, we prove that $F(T) \subset C_n$ for all $n \geq 1$. Indeed, $F(T) \subset C_1 = C$ is obvious. Suppose that $F(T) \subset C_k$ for some k. Then, for all $w \in F(T) \subset C_k$, we have

$$\begin{aligned}
\phi(w, y_k) &= \phi(w, J^{-1}(\alpha_k Jx_k + (1-\alpha_k)JTx_k)) \\
&= \|w\|^2 - 2\langle w, \alpha_k Jx_k + (1-\alpha_k)JTx_k\rangle \\
&\quad + \|\alpha_k Jx_k + (1-\alpha_k)JTx_k\|^2 \\
&\leq \|w\|^2 - 2\alpha_k\langle w, Jx_k\rangle - 2(1-\alpha_k)\langle w, JTx_k\rangle \\
&\quad + \alpha_k\|x_k\|^2 + (1-\alpha_k)\|Tx_k\|^2 \\
&= \alpha_k\phi(w, x_k) + (1-\alpha_k)\phi(w, Tx_k) \\
&\leq \alpha_k\phi(w, x_k) + (1-\alpha_k)\phi(w, x_k) \\
&= \phi(w, x_k),
\end{aligned}$$

which shows that $w \in C_{k+1}$. This implies that $F(T) \subset C_n$ for all $n \geq 1$. From $x_n = \Pi_{C_n}x_0$, we see that

$$\langle x_n - z, Jx_0 - Jx_n\rangle \geq 0, \quad \forall z \in C_n. \tag{2.2}$$

Since $F(T) \subset C_n$ for all $n \geq 1$, we have

$$\langle x_n - w, Jx_0 - Jx_n\rangle \geq 0, \quad \forall w \in F(T). \tag{2.3}$$

On the other hand, it follows from Lemma 1.2 that

$$\phi(x_n, x_0) = \phi(\Pi_{C_n}x_0, x_0) \leq \phi(w, x_0) - \phi(w, x_n) \leq \phi(w, x_0)$$

for each $w \in F(T) \subset C_n$ and for all $n \geq 1$. This shows that the sequence $\phi(x_n, x_0)$ is bounded. From (1.5), we see that the sequence $\{x_n\}$ is also bounded. Since the space is reflexive, we may, without loss of generality, assume that $x_n \rightharpoonup x$. Note that C_n is closed and convex for each $n \geq 1$. It is easy to see that $x \in \Omega$, where $\Omega = \bigcap_{n=0}^{\infty} C_n$. On the other hand, we have

$$\phi(x_n, x_0) \leq \phi(x_{n+1}, x_0) \leq \phi(x, x_0).$$

It follows that

$$\phi(x, x_0) \leq \liminf_{n\to\infty} \phi(x_n, x_0) \leq \limsup_{n\to\infty} \phi(x_n, x_0) \leq \phi(x, x_0).$$

This implies that

$$\lim_{n\to\infty} \phi(x_n, x_0) = \phi(x, x_0).$$

Hence, we have $\|x_n\| \to \|x\|$ as $n \to \infty$. In view of the Kadec-Klee property of E, we obtain that $x_n \to x$ as $n \to \infty$.

Next, we show that $x \in F(T)$. By the construction of C_n, we have that $C_{n+1} \subset C_n$ and $x_{n+1} = \Pi_{C_{n+1}} x_0 \in C_n$. It follows that

$$\begin{aligned} \phi(x_{n+1}, x_n) &= \phi(x_{n+1}, \Pi_{C_n} x_0) \\ &\leq \phi(x_{n+1}, x_0) - \phi(\Pi_{C_n} x_0, x_0) \\ &= \phi(x_{n+1}, x_0) - \phi(x_n, x_0). \end{aligned} \tag{2.4}$$

Letting $n \to \infty$ in (2.4), we obtain that $\phi(x_{n+1}, x_n) \to 0$. In view of $x_{n+1} \in C_{n+1}$, we arrive at $\phi(x_{n+1}, y_n) \leq \phi(x_{n+1}, x_n)$. It follows that

$$\lim_{n\to\infty} \phi(x_{n+1}, y_n) = 0. \tag{2.5}$$

From (1.6), we see that

$$\|y_n\| \to \|x\| \quad \text{as } n \to \infty. \tag{2.6}$$

It follows that

$$\|Jy_n\| \to \|Jx\| \quad \text{as } n \to \infty. \tag{2.7}$$

This implies that $\{Jy_n\}$ is bounded. Note that E is reflexive and E^* is also reflexive. We may assume that $Jy_n \rightharpoonup x^* \in E^*$. In view of the reflexivity of E, we see that $J(E) = E^*$. This shows that there exists an $\bar{x} \in E$ such that $J\bar{x} = x^*$. It follows that

$$\begin{aligned}\phi(x_{n+1}, y_n) &= \|x_{n+1}\|^2 - 2\langle x_{n+1}, Jy_n\rangle + \|y_n\|^2 \\ &= \|x_{n+1}\|^2 - 2\langle x_{n+1}, Jy_n\rangle + \|Jy_n\|^2.\end{aligned}$$

Taking $\liminf_{n\to\infty}$ the both sides of equality above yields that

$$\begin{aligned}0 &\geq \|x\|^2 - 2\langle x, x^*\rangle + \|x^*\|^2 = \|x\|^2 - 2\langle x, J\bar{x}\rangle + \|J\bar{x}\|^2 \\ &= \|x\|^2 - 2\langle x, J\bar{x}\rangle + \|\bar{x}\|^2 = \phi(x, \bar{x}).\end{aligned}$$

That is, $x = \bar{x}$, which in turn implies that $x^* = Jx$. It follows that $Jy_n \rightharpoonup Jx \in E^*$. Since (2.7) and E^* enjoys the Kadec-Klee property, we obtain that

$$Jy_n - Jx \to 0 \quad \text{as } n \to \infty.$$

Note that $J^{-1} : E^* \to E$ is demi-continuous. It follows that $y_n \rightharpoonup x$. Since (2.6) and E enjoys the Kadec-Klee property, we obtain that

$$y_n \to x \quad \text{as } n \to \infty.$$

Note that

$$\|x_n - y_n\| \leq \|x_n - x\| + \|x - y_n\|.$$

It follows that

$$\lim_{n\to\infty} \|x_n - y_n\| = 0. \tag{2.8}$$

Since J is uniformly norm-to-norm continuous on any bounded sets, we have

$$\lim_{n\to\infty} \|Jx_n - Jy_n\| = 0. \tag{2.9}$$

On the other hand, from the definition of y_n, we have

$$\|Jy_n - Jx_n\| = (1 - \alpha_n)\|JTx_n - Jx_n\|.$$

By the assumption on $\{\alpha_n\}$ and (2.9), we see that

$$\lim_{n\to\infty} \|Jx_n - JTx_n\| = 0. \tag{2.10}$$

On the other hand, noting that $J : E \to E^*$ is demi-continuous, we have $Jx_n \rightharpoonup Jx \in E^*$. In view of

$$|\|Jx_n\| - \|Jx\|| = |\|x_n\| - \|x\|| \le \|x_n - x\| \to 0 \quad \text{as } n \to \infty,$$

we arrive at $\|Jx_n\| \to \|Jx\|$ as $n \to \infty$. By virtue of the Kadec-Klee property of E^*, we obtain that $\|Jx_n - Jx\| \to 0$ as $n \to \infty$. In view of (2.10), we arrive at $\|JTx_n - Jx\| \to 0$ as $n \to \infty$. Since $J^{-1} : E^* \to E$ is demi-continuous, we have $Tx_n \rightharpoonup x$. Note that

$$|\|Tx_n\| - \|x\|| = |\|JTx_n\| - \|Jx\|| \le \|JTx_n - Jx\| \to 0 \quad \text{as } n \to \infty.$$

This implies that $\|Tx_n\| \to \|x\|$ as $n \to \infty$. From the Kadec-Klee property of E, we obtain that $\|Tx_n - x\| \to 0$ as $n \to \infty$. It follows from the closedness of T that $Tx = x$.

Finally, we show that $x = \Pi_{F(T)}x_0$. From $x_n = \Pi_{C_n}x_0$, we have

$$\langle x_n - w, Jx_0 - Jx_n\rangle \ge 0, \quad \forall w \in F(T) \subset C_n. \tag{2.11}$$

Taking the limit as $n \to \infty$ in (2.11), we obtain that

$$\langle x - w, Jx_0 - Jx\rangle \ge 0, \quad \forall w \in F(T),$$

and hence $x = \Pi_{F(T)}x_0$ by Lemma 1.1. This completes the proof. □

REMARK 2.1. Theorem 2.1 is a Banach version of Theorem 1.2.

REMARK 2.2. Theorem 2.1 improves the Theorem 1.3 in the following senses.

(1) From the computation point of view, the algorithm is more simple than the one considered in Theorem 1.3, that is, we remove the set "W_n".

(2) We extend the mapping from relatively nonexpansive mappings to quasi-ϕ-nonexpansive mappings, that is, we remove the restriction $F(T) = \widetilde{F}(T)$.

(3) We extend the space from uniformly smooth and uniformly convex Banach spaces to uniformly smooth and strictly convex Banach spaces which enjoy the Kadec-Klee property (note that every uniformly convex Banach spaces enjoy the Kadec-Klee property).

In the real Hilbert spaces, Theorem 2.1 is reduced to the following result.

COROLLARY 2.2. *Let C be a nonempty closed convex subset of a real Hilbert space H and let $T : C \to C$ be a closed quasi-nonexpansive mapping. Let $\{x_n\}$ be a sequence generated in the following manner:*

$$\begin{cases} x_0 \in H \quad \textit{chosen arbitrarily}, \\ C_1 = C, \\ x_1 = P_{C_1} x_0, \\ y_n = \alpha_n x_n + (1 - \alpha_n) T x_n, \\ C_{n+1} = \{z \in C_n : \|z - y_n\| \leq \|z - x_n\|\}, \\ x_{n+1} = P_{C_{n+1}} x_0, \quad \forall n \geq 0. \end{cases}$$

If the control sequence $\{\alpha_n\}$ satisfies the restrictions: $0 \leq \alpha_n < 1$ and $\limsup_{n\to\infty} \alpha_n < 1$, then $\{x_n\}$ converges strongly to $\Pi_{F(T)} x_0$.

REMARK 2.3. Theorem 2.1 improves the Theorem 1.1 in the following senses.

(1) From the computation point of view, the algorithm is more simple than the one considered in Theorem 1.1, that is, we remove the set "Q_n".

(2) We extend the mapping from nonexpansive mappings to quasi-nonexpansive mappings.

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