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A CORRECTION OF KELLEY'S PROOF ON THE EQUIVALENCE BETWEEN THE TYCHONOFF PRODUCT THEOREM AND THE AXIOM OF CHOICE

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Abstract. The Tychonoff product theorem is one of the most fundamental theorems in general topology. As is well-known, the proof of the Tychonoff product theorem relies on the axiom of choice. The converse was also conjectured by S. Kakutani and Kelley [1] then resolved this conjecture in his historical short note on 1950. However, the original proof due to Kelley has a flaw. According to this observation, we provide a correction of the proof in this paper.

1. Introduction

The Tychonoff product theorem which states that the Cartesian product of compact topological spaces is compact is one of the most fundamental theorems in general topology. It plays a central role in the development of a wealth of theorems within topology and applications of topology to other fields: the constructin of the Stone-Čech compactification, the Ascoli's theorem on the compactness of function spaces, and the proof of compactness of the maximal ideal space of a Banach algebra, etc. The theorem says that the Cartesian product of compact topological spaces is compact. As is well-known, the proof of the Tychonoff product theorem relies on the axiom of choice. How about the converse? Does the Tychonoff product theorem imply the axiom of choice? This was first conjectured by S. Kakutani, and

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Kelley [1] then resolved this conjecture in his historical short note on 1950.

However, the original proof due to Kelley [1] really had a flaw even though it is quite elementary. According to this observation, to make the proof complete, we provide a correction of the proof. This is the purpose of this note. We first state the axiom of choice as follows:

"If $\{X_\lambda \mid \lambda \in \Lambda\}$ is an indexed family of nonempty sets, then the Cartesian product $\prod_{\lambda\in\Lambda} X_\lambda$ is nonempty."

Let us recall the following pointed out by Kelley [1]: In the absence of the axiom of choice, it is necessary to define 'finite'. A set is finite if it may be ordered so that every nonempty subset has both a first and a last element in the ordering. Then the axiom of choice for finite families of sets can be proved (see Tarski [2] for a full discussion).

2. Analysis of the Kelley's proof

The proof of Kelley [1] can be divided into six steps.

Step 1. Adjoin a single point, say A, to each of the set X_λ, and define $Y_\lambda = X_\lambda \cup \{A\}$.

Step 2. Assign the cofinite topology $T_\lambda = \{G_\lambda \mid {G_\lambda}^c \text{ is a finite subset of } Y_\lambda\} \cup \{\emptyset\}$ for Y_λ.

Step 3. For each $\lambda \in \Lambda$, let $Z_\lambda = {P_\lambda}^{-1}(X_\lambda) = X_\lambda \times \prod_{\eta\neq\lambda} Y_\eta$ where P_λ is the λth projection map. Then Y_λ is compact and the product space $Y = \prod_{\lambda\in\Lambda} Y_\lambda$ is compact by the Tychonoff product theorem.

Step 4. Assert that X_λ is closed in Y_λ, hence Z_λ is closed in Y.

Step 5. For any finite subset Ω of Λ, show that the finite intersection $\bigcap_{\lambda\in\Omega} Z_\lambda$ is nonempty using the finite axiom of choice.

Step 6. By the finite intersection property of the compact space Y, claim that the whole intersection $\bigcap_{\lambda\in\Lambda} Z_\lambda = \prod_{\lambda\in\Lambda} X_\lambda = X$ is nonempty, as desired.

But, Step 4 is not true, so the following Steps 5 and 6 are not valid.

Actually X_λ is open but not closed in Y_λ, Indeed, if X_λ were closed in Y_λ, then ${X_\lambda}^c = \{A\}$ is open in Y_λ, Hence X_λ should be a finite set. This is not the case if X_λ is an infinite set.
Now we show that Z_λ is not closed in Y. In fact, if Z_λ were closed in Y, then ${Z_\lambda}^c = \{A\} \times \prod_{\eta\neq\lambda} Y_\eta$ is open in Y. As each Y_λ contains the definite element A, ${Z_\lambda}^c$ is nonempty. Thus there exists a finite number of nonempty open subsets O_{λ_i} in Y_{λ_i} $(\lambda_i \in \Lambda,\ i = 1, 2, \cdots, n)$ such that

$$O_{\lambda_1} \times \cdots \times O_{\lambda_n} \times \prod_{\eta\neq\lambda_i} Y_\eta \subseteq \{A\} \times \prod_{\eta\neq\lambda} Y_\eta.$$

Then we have two possibilities.
Case 1. $\lambda = \lambda_i$ for some i.

We see that $\emptyset \neq O_\lambda \subseteq \{A\}$, hence $O_\lambda = \{A\}$. Since X_λ is not closed but open in Y_λ, $O_\lambda = \{A\} = {X_\lambda}^c$ is not open in Y_λ. This contradicts the fact that O_λ is open in Y_λ.
Case 2. $\lambda = \eta$ for some $\eta \neq \lambda_i$.

Then we get $Y_\lambda \subseteq \{A\}$, so $Y_\lambda = \{A\}$. Thus $X_\lambda = \emptyset$, which contradicts $X_\lambda \neq \emptyset$. Clearly $Z_\lambda = {P_\lambda}^{-1}(X_\lambda) = X_\lambda \times \prod_{\eta\neq\lambda} Y_\eta$ is open in Y. This completes our analysis.

3. Correction of the proof

We are in a position to give a correct proof of the theorem. First we assign another topology $\mathcal{T}_\lambda$ for Y_λ by defining $\mathcal{T}_\lambda = \{\emptyset,\ \{A\},\ X_\lambda,\ Y_\lambda\}$. Clearly the topological space Y_λ is compact because the number of all open sets is finite. Hence $Y = \prod_{\lambda\in\Lambda} Y_\lambda$ is compact by the Tychonofff product theorem. Note that X_λ is closed in Y_λ by the definition of the topology $\mathcal{T}_\lambda$. Thus $Z_\lambda = {P_\lambda}^{-1}(X_\lambda) = X_\lambda \times \prod_{\eta\neq\lambda} Y_\eta$ is closed in Y since the projection map P_λ is continuous. Now we are ready to adopt Steps 5 and 6 in the previous section. For any finite subset Ω

of Λ, we can choose, by the finite axiom of choice, $x_\lambda \in X_\lambda (\neq \emptyset)$ for $\lambda \in \Omega$, and set $x_\eta = A$ for $\eta \in \Lambda \setminus \Omega$. Then we have

$$\bigcap_{\lambda \in \Omega} Z_\lambda = \bigcap_{\lambda \in \Omega} {P_\lambda}^{-1}(X_\lambda) = \prod_{\lambda \in \Omega} X_\lambda \times \prod_{\eta \in \Lambda \setminus \Omega} Y_\eta \neq \emptyset.$$

This means that the family of closed subsets $\{Z_\lambda \mid \lambda \in \Lambda\}$ of the compact topological space $Y = \prod_{\lambda \in \Lambda} Y_\lambda$ has the finite intersection property, therefore the whole intersection

$$\bigcap_{\lambda \in \Lambda} Z_\lambda = \bigcap_{\lambda \in \Lambda} {P_\lambda}^{-1}(X_\lambda) = \prod_{\lambda \in \Lambda} X_\lambda \neq \emptyset,$$

which proves the axiom of choice.

Remark. The mistake in the Kelley's proof [1] resides in the assignment of the cofinite topology T_λ for Y_λ. However, the argument is still valid as long as we endow Y_λ with any topology having the property:

"X_λ is closed in Y_λ, Y_λ is compact."

We can simply consider another such topology $\mathcal{T}_\lambda = \{\emptyset,\ \{A\},\ Y_\lambda\}$.

References

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