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HILBERT 2-CLASS FIELD TOWERS OF IMAGINARY QUADRATIC FUNCTION FIELDS

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Abstract. In this paper, we prove that the Hilbert 2-class field tower of an imaginary quadratic function field $F = \mathrm{k}(\sqrt{D})$ is infinite if $r_2(\mathcal{C}(F)) = 4$ and exactly one monic irreducible divisor of D is of odd degree, except for one type of Rédei matrix of F. We also compute the density of such imaginary quadratic function fields F.

1. Introduction and statement of results

Let $\mathrm{k} = \mathbb{F}_q(T)$ be the rational function field over the finite field $\mathbb{F}_q$ and $\mathbb{A} = \mathbb{F}_q[T]$. Write ∞ for the prime of k associated to $(1/T)$, which is called the infinite prime of k. For a finite separable extension F of k, let $\mathcal{O}_F$ be the integral closure of $\mathbb{A}$ in F and H_F be the Hilbert class field of F with respect to $\mathcal{O}_F$ ([6]). Let ℓ be a prime number. Write $F_1^{(\ell)}$ for the Hilbert ℓ-class field of $F_0^{(\ell)} = F$ (i.e., $F_1^{(\ell)}$ is the maximal ℓ-extension of F inside H_F) and inductively, $F_{n+1}^{(\ell)}$ for the Hilbert ℓ-class field of $F_n^{(\ell)}$ for $n \geq 1$. Then we obtain a sequence of fields

$$F_0^{(\ell)} = F \subset F_1^{(\ell)} \subset \cdots \subset F_n^{(\ell)} \subset \cdots ,$$

which is called the *Hilbert ℓ-class field tower of F*. We say that the Hilbert ℓ-class field tower of F is *infinite* if $F_n^{(\ell)} \neq F_{n+1}^{(\ell)}$ for each $n \geq 0$. For any multiplicative abelian group A, write $r_\ell(A) = \dim_{\mathbb{F}_\ell}(A/A^\ell)$, which is called the the ℓ-rank of A. By Schoof's theorem ([7]), we know that Hilbert ℓ-class field tower of F is infinite if $r_\ell(\mathcal{C}(F))$ is greater than or equal to $2 + 2\sqrt{r_\ell(\mathcal{O}_F^*) + 1}$, where $\mathcal{C}(F)$ and $\mathcal{O}_F^*$ are the ideal class group and the group of units of $\mathcal{O}_F$, respectively.

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In classical case, it has been conjectured by Martinet [5] that the Hilbert 2-class field tower of imaginary quadratic field F is infinite if $r_2(\mathcal{C}(F)) = 4$, and this conjecture has been studied by many authors ([2, 3, 8]).

Assume that q is odd. By an *imaginary quadratic function field*, we mean a quadratic extension F of k in which ∞ ramifies. Let $\mathcal{P}$ be the set of all monic irreducible polynomials in $\mathbb{A}$. Let γ be a generator of $\mathbb{F}_q^*$. Then any imaginary quadratic function field F of k can be written uniquely as $F = \mathrm{k}(\sqrt{D})$ with $D = aP_1 \cdots P_t$, where $a \in \{1, \gamma\}$, $P_i \in \mathcal{P}$ for $1 \le i \le t$ and $\deg D$ is odd. By genus theory, $r_2(\mathcal{C}(F)) = t - 1$. Since $\mathcal{O}_F^* \cong \mathbb{F}_q^*$, $r_2(\mathcal{O}_F^*) = 1$, so the Hilbert 2-class field tower of F is infinite if $r_2(\mathcal{C}(F)) \ge 5$ (i.e. $t \ge 6$) by Schoof's theorem. Write $r_4(\mathcal{C}(F)) = r_2(\mathcal{C}(F)^2)$, which is called the 4-rank of $\mathcal{C}(F)$. In [1], it has been shown that the Hilbert 2-class field tower of F is infinite if $r_4(\mathcal{C}(F)) \ge 3$, except some cases. Let M_F be the Rédei matrix associated to F (cf. §2.1). In this paper, we study the case where $r_2(\mathcal{C}(F)) = 4$ and exactly one monic irreducible polynomial of odd degree divides D, and prove that the Hilbert 2-class field tower of such a F is infinite, except for one type of Rédei matrix of F.

Theorem 1.1. *Let $F = \mathrm{k}(\sqrt{D})$ be an imaginary quadratic function field over* k. *Suppose that $r_2(\mathcal{C}(F)) = 4$ and exactly one monic irreducible divisor of D has odd degree, say $D = aP_1 \cdots P_5$ with $a \in \{1, \gamma\}$ and $\deg P_1$ is odd. Then the Hilbert 2-class field tower of F is infinite, except the case where*

$$M_F = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \end{pmatrix},$$

by changing the order of P_i's ($2 \le i \le 5$). In the exceptional case, $r_4(\mathcal{C}(F)) = 0$.

Now we compute the density of imaginary quadratic function fields $F = \mathrm{k}(\sqrt{D})$ satisfying Theorem 1.1 in all such ones. Write $\mathbf{A}$ for the set of all imaginary quadratic function fields $F = \mathrm{k}(\sqrt{D})$ with $r_2(\mathcal{C}(F)) = 4$ and exactly one monic irreducible divisor of D has odd degree. For any positive odd integer n, write $\mathbf{A}_n$ for the set of $F = \mathrm{k}(\sqrt{D}) \in \mathbf{A}$ with $\deg D = n$ and $\mathbf{A}_n^*$ for the subset of $\mathbf{A}_n$ consisting of $F = \mathrm{k}(\sqrt{D}) \in \mathbf{A}_n$ satisfying Theorem 1.1. Also we define a density

$$\delta = \lim_{\substack{n \to \infty \\ n:\mathrm{odd}}} \frac{|\mathbf{A}_n^*|}{|\mathbf{A}_n|}.$$

Then we have the following:

THEOREM 1.2. $\delta = 1 - 2^{-10} > 0.9990234$.

Theorem 1.2 says that most of imaginary quadratic function fields $F = \mathrm{k}(\sqrt{D})$ with $r_2(\mathcal{C}(F)) = 4$ and exactly one monic irreducible divisor of D has odd degree have infinite Hilbert 2-class field towers.

2. Preliminaries

2.1. 4-rank of $\mathcal{C}(F)$ and Rédei matrix M_F

Let $F = \mathrm{k}(\sqrt{D})$ be an imaginary quadratic function field with $D = aP_1 \cdots P_t$, where $a \in \{1, \gamma\}$, $P_i \in \mathcal{P}$ for $1 \le i \le t$ and $\deg D$ is odd. For simplicity, we assume that $\deg P_1$ is odd and $\deg P_i$ is even for $2 \le i \le t$. Let $M_F = (e_{ij})$ be the $t \times t$ matrix over $\mathbb{F}_2$ defined as follows: for $1 \le i \ne j \le t$, let $e_{ij} \in \mathbb{F}_2$ be defined by $(-1)^{e_{ij}} = (\frac{P_i}{P_j})$, and e_{ii} is defined to satisfy $\sum_{i=1}^{t} e_{ij} = 0$. This matrix M_F is called the Rédei matrix associated to F. Then 4-rank $r_4(\mathcal{C}(F))$ of $\mathcal{C}(F)$ satisfies the following equality ([9, §3]);

$$r_4(\mathcal{C}(F)) = t - 1 - \mathrm{rank}(M_F). \tag{2.1}$$

2.2. Martinet's inequality

For a finite extension K of k, write $S_\infty(K)$ for the set of primes of K lying over ∞. The following proposition is a special case of Theorem 2.1 in [1].

PROPOSITION 2.1. *Let E and K be finite (geometric) separable extensions of* k *such that E/K is a quadratic extension. Let $\gamma_{E/K}$ be the number of prime ideals of $\mathcal{O}_K$ that ramify in E and $\rho_{E/K}$ be the number of places $\mathfrak{p}_\infty$ in $S_\infty(K)$ that ramify or inert in E. Then the Hilbert 2-class field tower of E is infinite if*

$$\gamma_{E/K} \ge |S_\infty(K)| - \rho_{E/K} + 3 + 2\sqrt{2|S_\infty(K)| - \rho_{E/K} + 1}. \tag{2.2}$$

The inequality (2.2) in Proposition 2.1 is called Martinet's inequality. Let F be an imaginary quadratic function field over k. We remark that if there exists an extension E of F which has infinite Hilbert 2-class field tower and $F \subset E \subset F_1^{(2)}$, then F also has infinite Hilbert 2-class field tower. Applying the Martinet's inequality with above remark, we can prove the following corollary (see [1, §2] for details).

Corollary 2.2. *Let $F = \mathrm{k}(\sqrt{D})$ be an imaginary quadratic function field of* k. *If D has two distinct nonconstant monic divisors D' and D'' of even degrees satisfying $(\frac{D'}{Q_j}) = (\frac{D''}{Q_j}) = 1$ for monic irreducible divisors Q_j $(j = 1, 2)$ of D, then F has infinite Hilbert 2-class field tower.*

3. Proof of Theorems

3.1. Proof of Theorem 1.1

Let $F = \mathrm{k}(\sqrt{D})$ be an imaginary quadratic function field over k with $D = aP_1P_2P_3P_4P_5$, where $a \in \{1, \gamma\}$, $P_i \in \mathcal{P}$ for $1 \le i \le 5$ and $\deg D$ is odd. We also assume that $\deg P_1$ is odd and $\deg P_i$ is even for $2 \le i \le 5$. Then, by quadratic reciprocity law, the Rédei matrix M_F is symmetric.

First, suppose that there exists a column vector $\mathbf{m}_j = (e_{ij})$ of M_F for which at least two of e_{ij}'s $(2 \le i \le 5,\ i \ne j)$ are 0. Assuming that $e_{ij} = e_{kj} = 0$ with $i \ne j$ and $k \ne j$, put $K = \mathrm{k}(\sqrt{P_i}, \sqrt{P_k})$, which is a real biquadratic extension of k. Since $(\frac{P_i}{P_j}) = (\frac{P_k}{P_j}) = 1$, P_j splits completely in K. For any $l \ne i, j, k$, at least one of $(\frac{P_i}{P_l}), (\frac{P_k}{P_l})$ and $(\frac{P_iP_k}{P_l})$ is 1, so P_l splits into at least 2 primes in K. Applying Proposition 2.1 on E/K, where $E = F(\sqrt{P_i}, \sqrt{P_k})$, we see that the Hilbert 2-class field tower of E is infinite. Since E is contained in $F_1^{(2)}$, the Hilbert 2-class field tower of F is also infinite. In the following, we assume that at most one of e_{ij}'s $(2 \le i \le 5, i \ne j)$ is 0 for each column $\mathbf{m}_j = (e_{ij})$ for $1 \le j \le 5$ of M_F.

Case (i) Assume that one of e_{1i}'s $(2 \le i \le 5)$ is 0, say $e_{21} = 0$ and $e_{i1} = 1$ for $3 \le i \le 5$. If $e_{i2} = 1$ for $3 \le i \le 5$, put $K = \mathrm{k}(\sqrt{P_3P_4}, \sqrt{P_3P_5})$, so that P_1 splits completely in K. Since at least one of $(\frac{P_3P_4}{P_2}), (\frac{P_3P_5}{P_2})$ and $(\frac{P_4P_5}{P_2})$ is 1, P_2 splits into at least 2 primes in K. Applying Proposition 2.1, we see that the Hilbert 2-class field tower of F is infinite. On the other hand, if one of e_{i2}'s $(3 \le i \le 5)$ is 0, then we may assume that $e_{32} = 0$ and $e_{i2} = 1$ for $i = 4, 5$. Then $(\frac{P_2}{P_j}) = (\frac{P_4P_5}{P_j}) = 1$ for $j = 1, 3$, so the Hilbert 2-class field tower of F is infinite by Corollary 2.2.

Case (ii) Assume $e_{1i} = 1$ for $2 \le i \le 5$. If there is a column $\mathbf{m}_j = (e_{ij})$ $(2 \le j \le 5)$ of M_F satisfying $e_{ij} = 1$ for all i $(2 \le i \le 5, i \ne j)$, then $(\frac{P_kP_l}{P_1}) = (\frac{P_kP_m}{P_1}) = (\frac{P_kP_l}{P_j}) = (\frac{P_kP_m}{P_j}) = 1$ for $\{j, k, l, m\} = \{2, 3, 4, 5\}$, so the Hilbert 2-class field tower of F is infinite by Corollary 2.2. However, if there is no such column, we can't find an appropriate nonconstant monic divisors of D which satisfy the condition of Corollary 2.2. In this case, we have $e_{23} = e_{32} = e_{45} = e_{54} = 0$, by changing the order of P_i's. So, R_F is as described in the assertion of Theorem 1.1. This completes the proof of the Theorem 1.1.

3.2. Proof of Theorem 1.2

For any integers $n, t \geq 1$, write $\mathcal{P}(n)$ for the set of all monic irreducible polynomials of degree n, $\mathcal{P}(n,t)$ for the subset of $\mathcal{P}(n)$ consisting of monic square-free polynomials of degree n with t irreducible factors and $\mathcal{P}'(n,t)$ for the subset of $\mathcal{P}(n,t)$ consisting of $N = P_1 \cdots P_t$ with $\deg(P_i) \neq \deg(P_j)$ for $1 \leq i \neq j \leq t$. Then, as $n \to \infty$, we have

$$|\mathcal{P}(n)| = \frac{q^n}{n} + O\Big(\frac{q^{n/2}}{n}\Big), \tag{3.1}$$

$$|\mathcal{P}(n,t)| = \frac{q^n(\log n)^{t-1}}{(t-1)!n} + O\Big(\frac{q^n(\log n)^{t-2}}{n}\Big). \tag{3.2}$$

Let $\bar{\mathbf{A}}_n$ be the subset of $\mathbf{A}_n$ consisting of $F = \mathrm{k}(\sqrt{D}) \in \mathbf{A}_n$ with $D \in \mathcal{P}'(n,5)$, $\bar{\mathbf{A}}_n^* = \mathbf{A}_n^* \cap \bar{\mathbf{A}}_n$ and $\bar{\mathbf{B}}_n^* = \bar{\mathbf{A}}_n \setminus \bar{\mathbf{A}}_n^*$. By Proposition 2.2 in [10], we have

$$|\mathcal{P}(n,t) \setminus \mathcal{P}'(n,t)| = o\big(\frac{q^n(\log n)^{t-1}}{n}\big),$$

so

$$\delta = \lim_{\substack{n\to\infty \\ n:\mathrm{odd}}} \frac{2|\bar{\mathbf{A}}_n^*|}{2|\bar{\mathbf{A}}_n|} = \lim_{\substack{n\to\infty \\ n:\mathrm{odd}}} \frac{|\bar{\mathbf{A}}_n^*|}{|\bar{\mathbf{A}}_n|}. \tag{3.3}$$

We will compute the density $\lim_{\substack{n\to\infty \\ n:\mathrm{odd}}} \frac{|\bar{\mathbf{B}}_n^*|}{|\bar{\mathbf{A}}_n|}$. Using (3.1) with Lemma 3.5 in [10], we get

$$\begin{aligned} |\bar{\mathbf{A}}_n| &= \sum_{\substack{0<n_1<\cdots<n_5 \\ n_1\equiv 1(2), n_2\equiv\cdots\equiv n_5\equiv 0(2) \\ n_1+\cdots+n_5=n}} \sum_{P_1\in\mathcal{P}(n_1)} \sum_{P_2\in\mathcal{P}(n_2)} \cdots \sum_{P_5\in\mathcal{P}(n_5)} 1 \\ &= 2^{-4}\cdot\frac{q^n(\log n)^4}{4!n} + O\Big(\frac{q^n(\log n)^3}{n}\Big). \end{aligned} \tag{3.4}$$

We can compute the asymptotic formula of $|\bar{\mathbf{B}}_n^*|$ by using Proposition 2.3 in [4] as follows:

$$\begin{aligned} |\bar{\mathbf{B}}_n^*| &= \sum_{\substack{0<n_1<\cdots<n_5 \\ n_1\equiv 1(2), n_2\equiv\cdots\equiv n_5\equiv 0(2) \\ n_1+\cdots+n_5=n}} \sum_{P_1\in\mathcal{P}(n_1)} \sum_{\substack{P_2\in\mathcal{P}(n_2) \\ (\frac{P_1}{P_2})=-1}} \cdots \sum_{\substack{P_5\in\mathcal{P}(n_5) \\ 1\leq i\leq 3:(\frac{P_i}{P_5})=-1 \\ (\frac{P_4}{P_5})=1}} 1 \\ &= 2^{-14}\cdot\frac{q^n(\log n)^4}{4!n} + O\Big(\frac{q^n(\log n)^3}{n}\Big). \end{aligned} \tag{3.5}$$

From (3.4) and (3.5), we get $\lim_{\substack{n\to\infty\\ n:\mathrm{odd}}} \frac{|\bar{\mathbf{B}}_n^*|}{|\bar{\mathbf{A}}_n|} = 2^{-10}$, so $\delta = 1 - 2^{-10}$. This completes the proof of Theorem 1.2.

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