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# NEW EXISTENCE OF SOCIAL EQUILIBRIA IN GENERALIZED NASH GAMES WITH INSATIABILITY

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Abstract. In this paper, we first introduce a new model of strategic Nash game with insatiability, and next give two social equilibrium existence theorems for general strategic games which are comparable with the previous results due to Arrow and Debreu, Debreu, and Chang in several aspects.

## 1. Introduction

The classical results of Arrow and Debreu [1], Debreu [4], and Nash [8] have served as basic references for the existence of Nash equilibrium for non-cooperative strategic games. In all of them, convexity of strategy spaces, continuity and concavity of the payoff functions and the constraint correspondences were assumed. On the other hand, Shafer and Sonnenschein [10] extended the Debreu theorem on the existence of equilibrium in a generalized Nash game. Indeed, they maintained the sprits of the pioneering works of Arrow and Debreu [1] and Debreu [4]. Next, in 1976, Borglin and Keiding [2] first introduced the majorized concept of correspondences to obtain the existence of equilibrium in an abstract economy. Till now, there have been a number of generalizations, and also many applications of those theorems have been found in several areas, e.g., see [3,5,7,9] and references therein.

In this paper, we first introduce a new model of strategic Nash game with insatiability, and next give two new social equilibrium existence theorems for general strategic games by using Chang's maximal element existence theorem in [3], which are comparable with the previous results

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due to Arrow and Debreu [1], Chang [3], Debreu [4], and others in several aspects. Those results further generalize Nash's equilibrium existence theorem for a compact strategic game into a Hausdorff topological vector space with infinite players.

## 2. Preliminaries

If $A$ is a set, we shall denote by $2^A$ the family of all subsets of $A$. If $A$ is a subset of a vector space, we shall denote by $co\, A$ the convex hull of $A$. Let $E$ be a topological vector space and $A, X$ be nonempty subsets of $E$. If $S$, $T : A \to 2^E$ are correspondences (or multimaps), then $co\, T : A \to 2^E$ and $S \cap T : A \to 2^X$ are correspondences defined by $(co\, T)(x) = co\, T(x), (S \cap T)(x) = S(x) \cap T(x)$ for each $x \in A$, respectively.

Let $X$ be a nonempty subset of a topological vector space. A correspondence $\phi : X \to 2^X$ is said to be of *class* $L$ [5] if (i) for each $x \in X,\ x \notin co\, \phi(x)$, (ii) for each $y \in X,\ \phi^{-1}(y) = \{x \in X : y \in \phi(x)\}$ is open in $X$. Let $\phi : X \to 2^X$ be a given correspondence and $x \in X$ ; then a correspondence $\phi_x : X \to 2^X$ is said to be an *L-majorant of* $\phi$ *at* $x$ [5] if $\phi_x$ is of *class* $L$ and there exists an open neighborhood $N_x$ of $x$ in $X$ such that for each $z \in N_x,\ \phi(z) \subset \phi_x(z)$. The correspondence $\phi$ is said to be *L-majorized* if for each $x \in X$ with $\phi(x) \neq \emptyset$, there exists an $L$-majorant of $\phi$ at $x$.

Let $X$ be a nonempty subset of a topological vector space. A correspondence $\phi : X \to 2^X$ is said to have a *maximal element* $\bar{x} \in X$ if $\phi(\bar{x}) = \emptyset$. The existence of maximal element is essential in proving existence of equilibria in generalized games, and there have been numerous existence theorems on maximal elements in general settings by several authors, e.g., see [2,3,5,7,9].

In a recent paper [3], Chang proved a general maximal element existence theorem for $L_s$-majorized correspondences without assuming the local convexity of the given set $X_i$ nor the openness assumption of the set $\{x \in X \mid P_i(x) \neq \emptyset\}$. The following lemma is a special form of Theorem 2 in [3] by letting $K = D = \Pi_{i \in I} X_i$:

Lemma 2.1. *Let* $\Gamma = (X_i, P_i)_{i \in I}$ *be a qualitative game such that for each* $i \in I$,

*(1)* $X_i$ *is a nonempty compact convex subset of a Hausdorff topological vector space;*

*(2)* $P_i : X \to 2^{X_i}$ *is L-majorized.*

*Then there exists a maximal element* $\bar{x} \in X$, *i.e.* $P_i(\bar{x}) = \emptyset$ *for all* $i \in I$.

In compact convex settings, Chang remarked that Lemma 2.1 improves Theorem 3.4 of Kim and Yuan [7] without assuming that the set $G_i = \{x \in X \mid P_i(x) \neq \emptyset\}$ is open for each $i \in I$, which is very meaningful contribution because the openness assumption on $G_i$ is not natural nor easy to show.

The following is also an essential tool in proving the existence of equilibrium for generalized Nash game which is a special form of Theorem 4 in [3]:

LEMMA 2.2. *Let* $\Gamma = (X_i, A_i, P_i)_{i\in I}$ *be an abstract economy where* $I$ *is a (possibly uncountable) set of agents such that for each* $i \in I$,

*(1)* $X_i$ *is a nonempty compact convex subset of a Hausdorff topological vector space;*

*(2)* $A_i : X = \Pi_{i=1}^n X_i \to 2^{X_i}$ *is a correspondence such that each* $A_i(x)$ *is nonempty closed convex, and the set* $\mathcal{F}_i := \{x \in X \mid x_i \in A_i(x)\}$ *is closed in* $X$*;*

*(3) for each* $y \in X_i$, $A_i^{-1}(y)$ *is open in* $X$,

*(4)* $A_i \cap P_i : X \to 2^{X_i}$ *is L-majorized in the set* $\mathcal{F}_i$.

Then $\Gamma$ has an equilibrium choice $\hat{x} \in X$, i.e., for each $i \in I$,

$$\hat{x}_i \in A_i(\hat{x}) \quad \text{and} \quad A_i(\hat{x}) \cap P_i(\hat{x}) = \emptyset.$$

Throughout this paper, all topological spaces are assumed to be Hausdorff, and for the other standard notations and terminologies, we shall refer to [1-10].

## 3. A new model of generalized Nash game with insatiability

Now we recall some notions and terminologies in generalized Nash equilibrium for non-cooperative pure strategic games. Let the set $I$ of players be possibly uncountable. Then, a *generalized Nash game of normal form* (or *social system*) is the system of ordered triples $\Gamma = (X_i; T_i, f_i)_{i\in I}$, where for each player $i \in I$, the nonempty set $X_i$ is a player's pure strategy space, $T_i : X \to 2^{X_i}$ is a player's constraint correspondence, and $f_i : X \to \mathbb{R}$ is a player's payoff (or utility) function. The set $X$, *joint strategy space*, is the Cartesian product of the individual strategy spaces, and the element of $X_i$ is called a *strategy*. When $I$ is any set of players, we shall use the notation as

$$X_{-i} := \prod_{j \in I; j \neq i} X_j;$$

and hence we write a typical strategy profile $x = (x_i, x_{-i}) \in X = \Pi_{i \in I} X_i = X_i \times X_{-i}$. Then, a strategy profile $\bar{x} = (\bar{x}_i, \bar{x}_{-i}) \in X$ is called the *social equilibrium* (or *generalized Nash equilibrium*) for the generalized Nash game $\Gamma$ if the following system of inequalities holds: for each $i \in I$,

$$\bar{x}_i \in T_i(\bar{x}), \quad \text{and} \quad f_i(\bar{x}_i, \bar{x}_{-i}) \geq f_i(x_i, \bar{x}_{-i}) \quad \text{for each } x_i \in T_i(\bar{x}).$$

Next, we first introduce an economic condition which presents a kind of psychologic behavior as follows: Let $\Gamma = (X_i; T_i, f_i)_{i \in I}$ be a generalized Nash game with the set $I$ of players which is possibly infinite. Then we can consider an insatiable condition for the constraint correspondence $T_i : X \to 2^{X_i}$ which satisfies the following strict inequality with respect to the utility function $f_i$: let $x = (x_i, x_{-i}) \in X$ be an arbitrarily given profile. If $x_i \notin T_i(x)$, then there exists $y \in X_i$ such that $f_i(x_i, x_{-i}) < f_i(y, x_{-i})$.

The interpretation of this behavioral condition is as follow: If the strategy $x_i$ is not feasible in the game $\Gamma = (X_i; T_i, f_i)_{i \in I}$, the $i$-th player can not choose the strategy $x_i$ so that the value of utility function $f_i(x_i, x_{-i})$ can not be attainable. At this moment, the most of individual players might imagine and guess that there might be a better strategy $y \in X_i$ satisfying that $f_i(x_i, x_{-i}) < f_i(y, x_{-i})$, i.e., there might be a strategy having better payoff value. This is a kind of psychologic and natural economic sense in the real strategic game situation. Hence we formulate this concept as follows:

Definition 3.1. Let $\Gamma = (X_i; T_i, f_i)_{i \in I}$ be a generalized Nash game with the (possibly infinite) set $I$ of players. Then we call the constraint correspondence $T_i$ satisfy the *insatiable condition* (or *insatiability*) if the following condition holds: for any $x = (x_i, x_{-i}) \in X$,
$(*)$ if $x_i \notin T_i(x)$, then there exists $y \in X_i$ such that $f_i(x_i, x_{-i}) < f_i(y, x_{-i})$.

If the constraint correspondence $T_i$ satisfies the insatiable condition, then the strategy $x = (x_i, x_{-i}) \in X$ is a non-satiation strategy for the game $\Gamma$; and the contrapositive form of $(*)$ states that "if $f_i(x_i, x_{-i}) \geq f_i(y, x_{-i})$ for each $y \in X_i$, then $x_i \in T_i(x_i, x_{-i})$," so that the strategy $x_i \in X_i$ is clearly a best response for the player $i$. If for all player $i \in I$, this condition holds true for the same strategy $x = (x_i, x_{-i}) \in X$, then this action $x \in X$ is actually a social equilibrium for the generalized Nash game $\Gamma = (X_i; T_i, f_i)_{i\in I}$.

## 4. Existence of social equilibria in generalized Nash games

As an application of Lemma 2.1, we begin with a new existence theorem of social equilibrium for generalized Nash game with insatiability in a Hausdorff topological vector space.

THEOREM 4.1. *Let* $\Gamma = (X_i; T_i, f_i)_{i\in I}$ *be a generalized Nash game of normal form where* $X_i$ *is a nonempty compact convex subset in a Hausdorff topological vector space, and* $I$ *be any (possibly uncountable) set of players. Assume that for each* $i \in I$,

*(1)* $f_i : X = \Pi_{i\in I} X_i \to \mathbb{R}$ *is continuous, and quasiconcave in its i-th variable;*

*(2)* $T_i : X \to 2^{X_i}$ *is a constraint correspondence such that each* $T_i(x)$ *is a nonempty subset of* $X_i$, *and* $T_i$ *satisfies the insatiable condition.*

*Then the generalized Nash game* $\Gamma$ *has a social equilibrium* $\bar{x} \in X$, *i.e., for each* $i \in I$,

$$\bar{x}_i \in T_i(\bar{x}), \quad \textit{and} \quad f_i(\bar{x}_i, \bar{x}_{-i}) \geq f_i(x_i, \bar{x}_{-i}) \quad \textit{for all} \quad x_i \in T_i(\bar{x}).$$

*Proof.* For each $i \in I$, we first define a preference correspondence $P_i : X \to 2^X$ by for each $x \in X$,

$$P_i(x) := \{y \in X_i \mid f_i(x_i, x_{-i}) < f_i(y, x_{-i})\}.$$

If $x_i \notin T_i(x)$, then the insatiable assumption on $T_i$ means that $P_i(x)$ can not be an empty set, and $x_i \notin P_i(x)$. Next, we shall show that $P_i$ is a correspondence of class $L$.

For each $x \in X$, we will show that $x_i \notin co\, P_i(x)$. Suppose the contrary, i.e., there exists $x \in X$ such that $x_i \in co\, P_i(x)$. Then there exist $\lambda_1, \ldots, \lambda_n \in (0, 1]$, and $y_1, \ldots, y_n \in P_i(x)$ such that $x_i = \sum_{j=1}^{n} \lambda_j y_j$, and $f(x_i, x_{-i}) < f_i(y_j, x_{-i})$ for all $j = 1, \ldots, n$. Since $f_i$ is quasiconcave in its $i$-th variable, we have

$$f(x_i, x_{-i}) \;<\; \sum_{j=1}^{n} \lambda_j \cdot f_i(y_j, x_{-i}) \;\leq\; f_i\Big(\sum_{j=1}^{n} \lambda_j y_j, x_{-i}\Big) = f(x_i, x_{-i}),$$

which is a contradiction.

And, it is easy to see that $P_i^{-1}(y) = \{x \in X \mid f_i(x_i, x_{-i}) < f_i(y, x_{-i})\}$ is open in $X$ since $f_i : X = X_i \times X_{-i} \to \mathbb{R}$ is continuous in $X$. Hence $P_i$ is a correspondence of class $L$ for each $i \in I$.

Next, in order to apply Lemma 2.1, we shall introduce the qualitative game $\Gamma = (X_i, \phi_i)_{i \in I}$ as follows. For each $i \in I$, we introduce a set $\mathcal{F}_i$ of $T_i$ as

$$\mathcal{F}_i := \{x_i \in X_i \mid x_i \in T_i(x_i, x_{-i}) \text{ for } x = (x_i, x_{-i}) \in X\};$$

then $\mathcal{F}_i$ is a (possibly empty) subset of $X_i$.

Next, we define a correspondence $\phi_i : X \to 2^{X_i}$ by

$$\phi_i(x) = \begin{cases} T_i(x) \cap P_i(x), & \text{if } \quad x_i \in \mathcal{F}_i, \\ P_i(x), & \text{if } \quad x_i \notin \mathcal{F}_i. \end{cases}$$

Then we shall show that the qualitative game $\Gamma = (X_i, \phi_i)_{i \in I}$ has a maximal element. For this, we first show that $\phi_i$ is an $L$-majorized correspondence for each $i \in I$. Let $x \in X$ be arbitrarily given with $\phi_i(x) \neq \emptyset$. Since $P_i$ is a correspondence of class $L$, $\phi_i$ is clearly $L$-majorized since for any open neighborhood $N_x$ of $x$ in $X$, $\phi_i(z) \subseteq P_i(z)$ for each $z \in N_x$, and $P_i$ is of class $L$. Therefore, $\phi_i$ is an $L$-majorized correspondence for each $i \in I$.

Hence the qualitative game $\Gamma = (X_i, \phi_i)_{i \in I}$ satisfies the whole assumption of Lemma 2.1 so that there exists a maximal element $\bar{x} \in X$ such that $\phi_i(\bar{x}) = \emptyset$ for all $i \in I$. Then the maximal element $\bar{x} \in X$ is actually an equilibrium for the generalized Nash game $\Gamma$. Indeed, if there exists an $i \in I$ such that $\bar{x}_i \notin T_i(\bar{x})$, i.e., $\bar{x}_i \notin \mathcal{F}_i$; then $\phi_i(\bar{x}) = P_i(\bar{x}) = \{y \in X_i \mid f_i(\bar{x}_i, \bar{x}_{-i}) < f_i(y, \bar{x}_{-i})\}$. Since $T_i$ satisfies the insatiable condition so that $P_i(\bar{x})$ can not be an empty set. Hence we conclude that for each $i \in I$, $\bar{x}_i \in \mathcal{F}_i$, and so $\phi_i(\bar{x}) = T_i(x) \cap P_i(x) = \emptyset$; which means that for each $i \in I$,

$$\bar{x}_i \in T_i(\bar{x}), \quad \text{and} \quad f_i(\bar{x}_i, \bar{x}_{-i}) \geq f_i(x_i, \bar{x}_{-i}) \quad \text{for all} \quad x_i \in T_i(\bar{x}).$$

$\square$

Remark 4.2. (i) Theorem 4.1 has something different from the previous equilibrium existence theorems due to Debreu [4], Nash [8], Park [9] and others in the following aspects:

(a) the index set $I$ of players need not be finite;

(b) the strategy space $X_i$ need not be a subset of a locally convex space.

(c) $T_i$ need not be lower semicontinuous nor upper semicontinuous;

(d) the best response sets need not be contractible nor acyclic.

(ii) In Theorem 4.1, we shall need the insatiable condition which is a kind of behavioral psychologic assumption. It should be noted that the insatiable condition of $T_i$ is relatively easy to calculate and check the inequality on the payoff function $f_i$ in the generalized Nash game $\Gamma$; on the other hand, the assumptions in [4,9] are not easy to check the contractible or acyclic assumptions in general.

Finally, as an application of Lemma 2.2, we shall prove another existence theorem of social equilibrium for generalized Nash game in a Hausdorff topological vector space without assuming the insatiable condition for $T_i$:

THEOREM 4.3. *Let* $\Gamma = (X_i; T_i, f_i)_{i\in I}$ *be a generalized Nash game of normal form where* $X_i$ *is a nonempty compact convex subset in a Hausdorff topological vector space, and* $I$ *be any (possibly uncountable) set of players. Assume that*

*(1)* $f_i : X = \Pi_{i\in I} X_i \to \mathbb{R}$ *is continuous, and quasiconcave in its i-th variable;*

*(2) the constraint correspondence* $T_i : X \to 2^{X_i}$ *is such that each* $T_i(x)$ *is nonempty closed convex, and the set* $\mathcal{F}_i := \{x \in X \mid x_i \in T_i(x)\}$ *is closed in* $X$*;*

*(3) for each* $y \in X_i$, $T_i^{-1}(y)$ *is open in* $X$.

*Then the generalized Nash game* $\Gamma$ *has a social equilibrium* $\bar{x} \in X$.

*Proof.* The first part of proof is the same as the proof of Theorem 4.1. Indeed, for each $i \in I$, we first define a preference correspondence $P_i : X \to 2^{X_i}$ by for each $x \in X$,

$$P_i(x) := \{y \in X_i \mid f_i(x_i, x_{-i}) < f_i(y, x_{-i})\}.$$

Then $x_i \notin P_i(x)$ for each $x \in X$, and as shown in the proof of Theorem 4.1, we can obtain that $P_i$ is of class $L$. As remarked, note that $\mathcal{F}_i$ may be empty. And it is easy to see that $T_i \cap P_i : X \to 2^{X_i}$ is an $L$-majorized correspondence in the set $\mathcal{F}_i$. Indeed, for any $x \in \mathcal{F}_i$ with $(T_i \cap P_i)(x) \neq \emptyset$, and any open neighborhood $N_x$ of $x$ in $X$, $(T_i \cap P_i)(z) \subseteq P_i(z)$ for each $z \in N_x$, and $P_i$ is of class $L$ so that $T_i \cap P_i$ is $L$-majorized in $\mathcal{F}_i$. Therefore, the abstract economy $\widehat{\Gamma} = (X_i, T_i, P_i)_{i\in I}$ satisfies the whole assumption of Lemma 2.2 so that there exists an equilibrium point $\bar{x} \in X$ for the abstract economy $\widehat{\Gamma}$. Hence we conclude that for each $i \in I$, we have $\bar{x}_i \in T_i(\bar{x})$ and $T_i(x) \cap P_i(x) = \emptyset$, which means that for each $i \in I$,

$$\bar{x}_i \in T_i(\bar{x}), \quad \text{and} \quad f_i(\bar{x}_i, \bar{x}_{-i}) \geq f_i(x_i, \bar{x}_{-i}) \quad \text{for all} \quad x_i \in T_i(\bar{x}).$$

□

REMARK 4.4. When $T_i(x) := X_i$ for each $x \in X$ in Theorem 4.2, the assumptions (2) and (3) for the constraint correspondence $T_i$ is automatically satisfied so that we can obtain a generalization of Nash's equilibrium existence theorem in a Hausdorff topological vector space with infinite players.

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