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RESULTANT AND DISCRIMINANT OF LINEAR COMBINATION OF POLYNOMIALS

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Abstract. We investigate efficient method for the computation of resultant and discriminant of polynomials. The aim of this paper is to provide not only those values but also visual structure from elementary matrix calculation.

1. Introduction

Let K be a field of characteristic 0. The irreducibility of polynomial plays important roles more in recent application areas, such as coding, cryptography and computer algebra, etc. One of the main tools for determining irreducibility is discriminants and resultants. Resultant can provide a criterion whether a system of polynomials have a common root without explicitly solving for the roots [1]. Moreover resultants are applied systematically to provide constructive solutions to problems in computer graphic and algorithmic algebraic geometry ([4]).

In this paper we shall discuss the resultant and discriminant, and investigate effective computation methods involving matrices. There are some researches that focused on those computation ([3], [5]). Currently computer algebra systems such as Maple and Mathematica work well for those computations. The standard forms for discriminant and resultant of f and g in Maple are >discrim(f, x) and >resultant(f, g, x). However the aim of the work is to provide not only the value of resultant and discriminant but also the visual way from matrix calculation.

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2. Preliminary for resultant and discriminant

Let $f(x) = \sum_{i=0}^{n} a_i x^i$ and $g(x) = \sum_{j=0}^{m} b_j x^j$ be in $K[x]$ of degree n and m respectively. Let α_i $(1 \le i \le n)$ and β_j $(1 \le j \le m)$ be roots of $f(x)$ and $g(x)$ in some splitting fields. The resultant of f and g is defined by

$$R(f,g) = a_n^m b_m^n \prod_{i=1}^{n} \prod_{j=1}^{m} (\alpha_i - \beta_j).$$

If we write $f(x) = a_n \prod_{i=1}^{n}(x-\alpha_i)$ and $g(x) = b_m \prod_{j=1}^{m}(x-\beta_j)$ then

$$R(f,g) = a_n^m \prod_{i,j=1}^{n,m} b_m(\alpha_i - \beta_j) = a_n^m \prod_{i=1}^{n} g(\alpha_i) = (-1)^{nm} b_m^n \prod_{j=1}^{m} f(\beta_j).$$

We may refer to [2] and [6] as standard references for resultant. One of the most remarkable results about resultant is that $R(f,g)$ is described in terms of determinant of the Sylvester matrix of f and g.

Lemma 2.1. *Let $f(x) = \sum_{i=0}^{n} a_i x^i$ and $g(x) = \sum_{j=0}^{m} b_j x^j \in K[x]$. Then $R(f,g)$ is the determinant of $(n+m)\times(n+m)$ Sylvester matrix composed of all coefficients:*

$$Syl(f,g) = \begin{bmatrix} a_n & a_{n-1} & a_{n-2} & \cdots & a_1 & a_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n & a_{n-1} & \cdots & \cdots & a_0 \\ b_m & b_{m-1} & \cdots & b_1 & b_0 & 0 & \cdots & \cdots \\ \vdots & \vdots & \ddots & \cdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \cdots & b_m & b_{m-1} & \cdots & b_0 \end{bmatrix}$$

For $f(x) \in K[x]$ with leading coefficient a_n and roots α_i $(1 \le i \le n)$ in a splitting field of K, the discriminant Δ of f is defined by

$$\Delta(f) = a_n^{2n-2} \prod_{1 \le i < j \le n} (\alpha_i - \alpha_j)^2.$$

Lemma 2.2. *Let $f(x) \in K[x]$ be of degree n and $f'(x)$ be the formal derivative of $f(x)$. Then $\Delta(f) = (-1)^{n(n-1)/2} a_n^{-1}\, R(f,f')$.*

$\Delta(f) = 0$ if and only if f shares roots with its derivative, while $R(f,g) = 0$ if and only if f and g have at least one common root.

3. Resultant of the product of polynomials

We shall study the resultant of the product of polynomials. If $\deg f = n$ and $\deg g = m$, the resultant involves computations of $(n+m)\times(n+m)$ matrix which is too large to compute. Hence it is useful to decompose a polynomial into smaller degree ones as follow.

THEOREM 3.1. *Let $f(x), g(x), f_i(x)$ and $g_i(x) \in K[x]$. Then*

(1) $R(\underbrace{f\cdots f}_{s\ times}, g) = R(f,g)^s$ *and* $R(f^s, g^u) = R(f,g)^{su}$ *for* $s, u > 0$.

(2) $R(f\prod_i f_i, g\prod_j g_j) = R(f,g)\prod_j R(f,g_j)\prod_i R(f_i,g)\prod_{i,j} R(f_i,g_j)$.

Proof. Write $f(x) = a_n\prod_{i=1}^n (x-\alpha_i)$, $\quad g(x) = b_m\prod_{j=1}^m (x-\beta_j)$. It is known that $R(fh,g) = R(f,g)R(h,g)$ for any $h(x)\in K[x]$. In fact, if $h(x) = c_t\prod_{k=1}^t (x-\gamma_k)$ with roots γ_k of $h(x)$ then

$$(fh)(x) = a_n c_t \prod_i^n \prod_k^t (x-\alpha_i)(x-\gamma_k)$$

has zeros α_i and γ_k. By letting δ_i by $\delta_i = \alpha_i$ for $1\le i\le n$, and $\delta_{n+k} = \gamma_k$ for $1\le k\le t$, we have

$$\begin{aligned} R(fh,g) &= (a_n c_t)^m b_m^{n+t} \prod_{fh(\delta_i)=0,\ g(\beta_j)=0} (\delta_i - \beta_j) \\ &= a_n^m b_m^n \prod_{g(\beta_j)=0,\ i=1}^{n} (\alpha_i-\beta_j)\cdot c_t^m b_m^t \prod_{g(\beta_j)=0,\ k=1}^{t} (\gamma_k - \beta_j) \\ &= R(f,g)\cdot R(h,g). \end{aligned}$$

So $R(f\cdots f, g) = R(f,g)^s$ and $R(f^s,g^u) = R(f,g)^{su}$ for any $s,u>0$.

Moreover since $R(\prod f_i, g) = \prod R(f_i,g)$, we have

$$R(f\prod_i f_i, g\prod_j g_j) = R(f,g)R(f,\prod_j g_j)R(\prod_i f_i, g)R(\prod_i f_i, \prod_j g_j)$$

$$= R(f,g)\prod_j R(f,g_j)\prod_i R(f_i,g)\prod_{i,j} R(f_i,g_j). \ \square$$

For example,

$$\begin{aligned} & R(x^6+2x^4-x^2-2, x^2-2) \\ =\ & R(x^2+1,x^2-2)R(x^2-1,x^2-2)R(x^2+2,x^2-2) \\ =\ & f(\sqrt{2})f(-\sqrt{2})g(1)g(-1)h(\sqrt{2})h(-\sqrt{2}) = 3^2(-1)^2 4^2 = 12^2, \end{aligned}$$

where $f(x) = x^2+1$, $g(x) = x^2-2$ and $h(x) = x^2+2$.

Euclid algorithm says that $\gcd(n,m)$ $(n \geq m)$ equals that of m and the remainder of n by m. A similar algorithm holds for resultant that $R(f,g)$ equals that of g and the remainder of $f(x)$ by $g(x)$. In fact, if $\deg f = n \geq m = \deg g$ and $f(x) = q(x)g(x) + r(x)$ with $q(x)$, $r(x) \in K[x]$ $(r(x) = 0$ or $\deg(r) < m)$ then with roots β_j $(1 \leq j \leq m)$ of $g(x)$ in some splitting fields, we have

$$R(f,g) = (-1)^{nm} b_m^n \prod_{j=1}^{m} (q(\beta_j)g(\beta_j) + r(\beta_j)) = (-1)^{nm} b_m^n \prod_{j=1}^{m} r(\beta_j)$$

because $g(\beta_j) = 0$. Thus if we let $0 \leq \deg r = t < m$ then

$$\begin{aligned} R(f,g) &= (-1)^{nm}(-1)^{-mt} b_m^{n-t} (-1)^{mt} b_m^t \prod_j r(\beta_j) \\ &= (-1)^{m(n-t)} b_m^{n-t} (-1)^{mt} b_m^t \prod_j r(\beta_j) = (-1)^{m(n-t)} b_m^{n-t} R(r,g). \end{aligned}$$

Moreover $R(f, fh+g) = R(f,g)$ for any f, g and h.

Theorem 3.2. *Let $f(x), g(x), h(x)$, $f_i(x) \in K[x]$ and $s, u > 1$. Then $\Delta(\prod_{i=1}^s f_i) = \prod_{i=1}^s \Delta(f_i) \left(\prod_{1 \leq i < j \leq s} R(f_i, f_j)\right)^2$ and $\Delta(f^s g^u) = 0$.*

Proof. From the above consideration we have

$$R(fg, (fg)') = R(fg, f'g + fg') = R(f, f'g + fg') \cdot R(g, f'g + fg').$$

Moreover since

$$R(f, f'g + fg') = (-1)^{n^2(m-1)} \ a_n^{n(m-1)} \ R(f, f'g)$$

and $R(g, f'g + fg') = (-1)^{m^2(n-1)} \ b_m^{m(n-1)} \ R(g, fg')$, it follows that

$$\begin{aligned} &R(fg, (fg)') \\ &= (-1)^{n^2(m-1)}(-1)^{m^2(n-1)} a_n^{n(m-1)} b_m^{m(n-1)} R(f, f'g) R(g, fg') \\ &= (-1)^{n^2(m-1)+m^2(n-1)+nm} a_n^{n(m-1)} b_m^{m(n-1)} R(f, f') R(f,g)^2 R(g, g'), \end{aligned}$$

because $R(f,g) = (-1)^{nm} R(g,f)$. By Lemma 2.2 and Theorem 3.1,

$$\begin{aligned} &\Delta(fg) = (-1)^{nm(nm-1)/2} a_n^{-1} b_m^{-1} R(fg, (fg)') \\ &= (-1)^{nm(nm-1)/2} a_n^{-1} b_m^{-1} (-1)^{n^2(m-1)+m^2(n-1)+nm} \ a_n^{n(m-1)} b_m^{m(n-1)} \\ &\quad \cdot \ (-1)^{-n(n-1)/2} \ a_n \Delta(f) \ (-1)^{-m(m-1)/2} \ b_m \Delta(g) \cdot R(f,g)^2 \\ &= \Delta(f)\Delta(g)R(f,g)^2. \end{aligned}$$

Thus $\Delta(f^2) = \Delta(f)\Delta(f)R(f,f)^2 = 0$ for $R(f,f) = 0$. And $\Delta(f^t) = 0 = \Delta(g^s)$, so $\Delta(f^s g^u) = \Delta(f^s)\Delta(g^u) \ R(f^s, g^u) = 0$. Now for any $h(x)$,

$$\Delta(fgh) = \Delta(f) \ \Delta(g) \ \Delta(h) \ \left(R(f,g) \ R(f,h) \ R(g,h)\right)^2,$$

so $\Delta(\prod_{i=1}^s f_i) = \prod_{i=1}^s \Delta(f_i) (\prod_{1 \leq i < j \leq s} R(f_i, f_j))^2$ by induction. □

The example below Theorem 3.1 can be seen by Theorem 3.2:

$$R(x^6+2x^4-x^2-2, x^2-2)$$
$$= R((x^2-2)(x^4+4x^2+7)+12, x^2-2) = R(12, x^2-2) = 12^2.$$

In next theorem we study relationships between resultant of $f(x)$ and that of monic polynomial of $f(x)$. Indeed if $f(x) = a_n f_*(x)$ and $f'(x) = a_n {f_*}'(x)$ then according to some algebraic properties of resultant,

$$\begin{aligned} \Delta(f) &= (-1)^{n(n-1)/2} a_n^{-1}\ R(a_n, a_n)\ R(a_n, {f_*}')\ R(f_*, a_n)\ R(f_*, {f_*}') \\ &= a_n^{-1}\ a_n^{n-1} a_n^n\ (-1)^{n(n-1)/2} R(f_*, {f_*}')\ =\ a_n^{2n-2}\ \Delta(f_*), \end{aligned}$$

thus $R(f, g) = R(a_n, b_m)\ R(a_n, g_*)\ R(f_*, b_m)\ R(f_*, g_*) = a_n^m b_m^n R(f_*, g_*)$, and $R(f, f_*) = R(f, \frac{1}{a_n} f) = R(f, \frac{1}{a_n})\ R(f, f) = 0$.

This can be illustrated visually via matrix operation.

Theorem 3.3. *Let $f(x)$ and $g(x)$ be of degree n, m with leading coefficients a_n and b_m. Let $f_*(x) = a_n^{-1} f(x)$ and $g_*(x) = b_m^{-1} g(x)$. Then $R(f, g) = a_n^n b_m^m R(f_*, g_*)$ and $\Delta(f) = a_n^{2n-2} \Delta(f_*)$.*

Proof. Without loss of generality, let $f_*(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$ and $f(x) = a_n f_*(x)$. And let $g_*(x) = x^m + b_{m-1}x^{m-1} + \cdots + b_0$ and $g(x) = b_m g_*(x)$. Then

$$R(f,g) = \begin{vmatrix} a_n & a_n a_{n-1} & \cdots & a_n a_0 & 0 & 0 & \cdots \\ & & \ddots & & & \ddots & \\ 0 & 0 & \cdots & a_n & a_n a_{n-1} & \cdots & a_n a_0 \\ b_m & b_m b_{m-1} & \cdots & \cdots & b_m b_0 & 0 & \cdots \\ & & \ddots & & & \ddots & \\ 0 & 0 & 0 & \cdots & b_m & b_m b_{m-1} & b_m b_0 \end{vmatrix}$$

Since the upper n rows contain a_n commonly while the lower m rows contain b_m,

$$R(f,g) = a_n^n b_m^m \begin{vmatrix} 1 & a_{n-1} & \cdots & a_0 & 0 & \cdots\ 0 \\ 0 & 1 & a_{n-1} & \cdots & a_0 & \cdots\ 0 \\ & & \ddots & & & \ddots \\ 0 & 0 & \cdots & 1 & a_{n-1} & \cdots\ a_0 \\ 1 & b_{m-1} & \cdots & \cdots & b_0 & \cdots\ 0 \\ & & & \ddots & & \ddots \\ 0 & 0 & 0 & 1 & \cdots & b_{m-1}\ b_0 \end{vmatrix} = a_n^n b_m^m R(f_*, g_*).$$

Moreover since $R(f, f') = a_n^n a_n^{n-1} R(f_*, f_*') = a_n^{2n-1} R(f_*, f_*')$, we have

$$\Delta(f) = (-1)^{n(n-1)/2} a_n^{-1}\ a_n^{2n-1}\ R(f_*, f_*') = a_n^{2(n-1)} \Delta(f_*).$$

□

Theorem 3.3 guarantee that only monic polynomial is enough to be considered for resultants and discriminants.

If we know all roots α_i of $f(x)$ and β_j of $g(x)$ explicitly then

$$R(f,g) = a_n^m b_m^n \prod_{i,j}(\alpha_i - \beta_j) = (-1)^{nm} b_m^n \prod_j f(\beta_j) = a_n^m \prod_i g(\alpha_i).$$

For instance, if $f(x) = x^2 - 1$ then $R(f,g) = g(1)g(2)$ for any $g(x)$. If α_1 and α_2 are roots of $f(x) = ax^2 + bx + c$ then

$$R(f,g) = a^{\deg g} g(\alpha_1)g(\alpha_2) \text{ and } \Delta(f) = a^2(\alpha_1 - \alpha_2)^2 = b^2 - 4ac.$$

However finding roots of polynomials is not an easy task in general. Instead, the discriminant is used to get information about the roots of polynomials.

4. Resultant of the linear combination of polynomials

The multiplicative equality $R(fh,g) = R(f,g)R(h,g)$ was discussed in Section 3. But the additive equality $R(f+h,g) = R(f,g) + R(h,g)$ does not hold that, if $f(x) = h(x) = x+1$, $g(x) = x^2 + x + 1$ then

$$R(f+h,g) = 2\begin{vmatrix} 2 & 2 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 2\left(\begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix}\right)$$

$$= 2\ (R(f,g) + R(h,g)) \ \neq\ R(f,g) + R(h,g).$$

However $R(f, g+g) = 2R(f,g)$ if $\deg f = 1$. And there are also many situations that the equality holds.

THEOREM 4.1. *Let $g(x) = b_1x + b_0 \in K[x]$. If $\deg f = n$ and $degh = t$ with $n \geq t$ then $R(f+h,g) = R(f,g) + (-b_1)^{n-t}R(h,g)$.*

Proof. Let $f(x) = \sum_{i=0}^n a_i x^i$ and $h(x) = \sum_{i=0}^t c_i x^i$. Since $n \geq t$, $(f+h)(x) = \sum_{i=t+1}^n a_i x^i + \sum_{i=0}^t (a_i + c_i)x^i$, so

$$R(f+h,g) = \begin{vmatrix} a_n & a_{n-1} & \cdots & a_{t+1} & a_t + c_t & \cdots & a_0 + c_0 \\ b_1 & b_0 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & b_1 & b_0 & 0 & \cdots & \cdots & 0 \\ \\ 0 & 0 & \cdots & \cdots & \ddots & b_1 & b_0 \end{vmatrix}$$

$$= \begin{vmatrix} a_n\ a_{n-1} & \cdots & a_{t+1} & a_t & \cdots & a_0 \\ b_1\ b_0 & 0 & \cdots & 0 & \cdots & 0 \\ 0\ b_1 & b_0 & 0 & \cdots & \cdots & 0 \\ \\ 0\ 0 & 0 & \cdots & \ddots & b_1 & b_0 \end{vmatrix} + \begin{vmatrix} 0\ 0 & \cdots & 0 & c_t & \cdots & c_0 \\ b_1\ b_0 & 0 & \cdots & 0 & \cdots & 0 \\ 0\ b_1 & b_0 & 0 & \cdots & \cdots & 0 \\ \\ 0\ 0 & 0 & \cdots & \ddots & b_1 & b_0 \end{vmatrix}$$

$$= R(f,g) + (-b_1) \begin{vmatrix} 0 & \cdots & 0 & c_t & \cdots & c_0 \\ b_1 & b_0 & 0 & \cdots & \cdots & 0 \\ 0 & b_1 & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \cdots & \cdots & b_1 & b_0 \end{vmatrix} = \cdots$$

$$= R(f,g) + (-b_1)^{n-t} \begin{vmatrix} c_t & \cdots & c_1 & c_0 \\ b_1 & b_0 & \cdots & 0 \\ 0 & \ddots & b_1 & b_0 \end{vmatrix} = R(f,g) + (-b_1)^{n-t} R(h,g).$$

□

THEOREM 4.2. *Let $degf = n$, $degg = m$, and b_m be the leading coefficient of g. If $n = m$ then $R(f+g, f) = R(g, f) = (-1)^{nm} R(f, g)$. If $n < m$ then $R(f+g, f) = R(g, f)$ and $R(f+g, g) = b_m^{m-n}(-1)^{nm} R(f, g)$.*

Proof. Let $f(x) = \sum_{i=0}^{n} a_i x^i$ and $g(x) = \sum_{i=0}^{m} b_i x^i$. If $n = m$ then $(f+g)(x) = \sum_{i=0}^{m} (a_i + b_i) x^i$ and $R(f+g, f)$ is the determinant of the $2n \times 2n$ matrix:

$$R(f+g,f) = \begin{vmatrix} a_n + b_n & \cdots & a_0 + b_0 & \cdots & 0 \\ & \ddots & & \ddots & \\ 0 & \cdots & a_n + b_n & \cdots & a_0 + b_0 \\ a_n & \cdots & a_0 & \cdots & 0 \\ 0 & \ddots & a_n & \ddots & a_0 \end{vmatrix}$$

$$= \begin{vmatrix} b_n & \cdots & \cdots & b_0 & 0 & \cdots & 0 \\ 0 & 0 & \ddots & b_n & \cdots & \ddots & b_0 \\ a_n & \cdots & \cdots & a_0 & 0 & \cdots & 0 \\ 0 & 0 & \ddots & a_n & \cdots & \ddots & a_0 \end{vmatrix} = R(g,f)$$

where we subtract $n + i$th row from ith row for $1 \leq i \leq n$.

Before we go on, we begin with an example for $n = 3 < m = 5$. Then

$$R(f+g,f) = \begin{vmatrix} b_5 & b_4 & a_3+b_3 & a_2+b_2 & a_1+b_1 & 0 & 0 \\ 0 & b_5 & b_4 & a_3+b_3 & a_2+b_2 & \cdots & 0 \\ 0 & 0 & b_5 & b_4 & a_3+b_3 & \cdots & a_0+b_0 \\ a_3 & a_2 & a_1 & a_0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \ddots & a_3 & \cdots & a_0 \end{vmatrix}$$

$$= \begin{vmatrix} b_5 & b_4 & b_3 & b_2 & b_1 & b_0 & 0 & 0 \\ 0 & b_5 & b_4 & b_3 & b_2 & b_1 & b_0 & 0 \\ 0 & 0 & b_5 & b_4 & b_3 & b_2 & b_1 & b_0 \\ a_3 & a_2 & a_1 & a_0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \ddots & a_3 & a_2 & a_1 & a_0 \end{vmatrix} = R(g,f)$$

by subtracting $m + i$th row from ith for $1 \leq i \leq n$. On the other hand,

$$R(f+g,g)=\begin{vmatrix} b_5 & b_4 & a_3+b_3 & \cdots & a_0+b_0 & 0 & \cdots \\ 0 & b_5 & b_4 & a_3+b_3 & \cdots & a_0+b_0 & \cdots \\ 0 & 0 & 0 & 0 & b_5 & b_4 & \cdots \\ b_5 & b_4 & b_3 & \cdots & b_0 & 0 & \cdots \\ 0 & 0 & 0 & \ddots & b_5 & b_4 & \cdots \end{vmatrix}$$

$$=(-b_5)^2\begin{vmatrix} a_3 & a_2 & a_1 & a_0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 & a_3 & a_2 & a_1 & a_0 \\ b_5 & b_4 & b_3 & b_2 & b_1 & b_0 & 0 & 0 \\ 0 & b_5 & b_4 & b_3 & b_2 & b_1 & b_0 & 0 \\ 0 & 0 & b_5 & b_4 & b_3 & b_2 & b_1 & b_0 \end{vmatrix}=b_5^2\cdot R(f,g).$$

Now for any $\deg f=n<\deg g=m$, we have

$$R(f+g,f)=\begin{vmatrix} b_m & \cdots & a_n+b_n & \cdots & a_1+b_1 & a_0+b_0 & 0 \\ 0 & \ddots & b_m & \cdots & a_n+b_n & \ddots & a_0+b_0 \\ a_n & \cdots & a_1 & a_0 & 0 & \cdots & 0 \\ 0 & \ddots & 0 & a_n & \cdots & \ddots & a_0 \end{vmatrix}$$

$$=\begin{vmatrix} b_m & \cdots & b_1 & b_0 & 0 & \cdots \\ 0 & \ddots & b_m & \cdots & \ddots & b_0 \\ a_n & \cdots & a_1 & a_0 & 0 & \cdots \\ 0 & \ddots & 0 & a_n & \ddots & a_0 \end{vmatrix} = R(g,f),$$

by subtracting $m+i$th row from ith row for $1\le i\le n$. And similarly

$$R(f+g,g)=\begin{vmatrix} b_m & \cdots & a_n+b_n & \cdots & a_1+b_1 & \cdots & 0 \\ 0 & \ddots & b_m & \cdots & a_n+b_n & a_1+b_1 & a_0+b_0 \\ b_m & \cdots & b_n & \cdots & b_1 & \cdots & 0 \\ 0 & \ddots & b_m & \cdots & b_n & b_1 & b_0 \end{vmatrix}$$

$$=(-b_m)^{m-n}\begin{vmatrix} a_n & \cdots & a_1 & a_0 & \cdots & 0 \\ 0 & \ddots & a_n & \ddots & a_1 & a_0 \\ b_m & \cdots & \cdots & b_1 & b_0 & \cdots 0 \\ 0 & \ddots & b_m & \ddots & b_1 & b_0 \end{vmatrix} = (-b_m)^{m-n}R(f,g).$$

□

We study resultant $R(s_1f+s_2g,t_1f+t_2g)$ of linear combination of f and g. If $f(x)=a_1x+a_0$ and $g(x)=b_1x+b_0$ are linear then

$$R(s_1f+s_2g,t_1f+t_2g)=\begin{vmatrix} s_1a_1+s_2b_1 & s_1a_0+s_2b_0 \\ t_1a_1+t_2b_1 & t_1a_0+t_2b_0 \end{vmatrix}$$

$$=\begin{vmatrix} s_1a_1 & s_1a_0 \\ t_1a_1 & t_1a_0 \end{vmatrix}+\begin{vmatrix} s_1a_1 & s_1a_0 \\ t_2b_1 & t_2b_0 \end{vmatrix}+\begin{vmatrix} s_2b_1 & s_2b_0 \\ t_1a_1 & t_1a_0 \end{vmatrix}+\begin{vmatrix} s_2b_1 & s_2b_0 \\ t_2b_1 & t_2b_0 \end{vmatrix}$$

$$= s_1t_2\begin{vmatrix} a_1 & a_0 \\ b_1 & b_0 \end{vmatrix}-s_2t_1\begin{vmatrix} a_1 & a_0 \\ b_1 & b_0 \end{vmatrix}=\begin{vmatrix} s_1 & t_1 \\ s_2 & t_2 \end{vmatrix}R(f,g).$$

Now if $f(x) = a_2x^2 + a_1x + a_0$ and $g(x) = b_2x^2 + b_1x + b_0$ are quadratic, then

$$R(s_1f+s_2g, t_1f+t_2g) = \begin{vmatrix} s_1a_2+s_2b_2 & s_1a_1+s_2b_1 & s_1a_0+s_2b_0 & 0 \\ 0 & s_1a_2+s_2b_2 & s_1a_1+s_2b_1 & s_1a_0+s_2b_0 \\ t_1a_2+t_2b_2 & t_1a_1+t_2b_1 & t_1a_0+t_2b_0 & 0 \\ 0 & t_1a_2+t_2b_2 & t_1a_1+t_2b_1 & t_1a_0+t_2b_0 \end{vmatrix}$$

$$= s_1^2 \begin{vmatrix} a_2 & a_1 & a_0 & 0 \\ 0 & a_2 & a_1 & a_0 \\ t_1a_2+t_2b_2 & t_1a_1+t_2b_1 & t_1a_0+t_2b_0 & 0 \\ 0 & t_1a_2+t_2b_2 & t_1a_1+t_2b_1 & t_1a_0+t_2b_0 \end{vmatrix}$$

$$+ s_1s_2 \begin{vmatrix} a_2 & a_1 & a_0 & 0 \\ 0 & b_2 & b_1 & b_0 \\ t_1a_2+t_2b_2 & t_1a_1+t_2b_1 & t_1a_0+t_2b_0 & 0 \\ 0 & t_1a_2+t_2b_2 & t_1a_1+t_2b_1 & t_1a_0+t_2b_0 \end{vmatrix}$$

$$+ s_1s_2 \begin{vmatrix} b_2 & b_1 & b_0 & 0 \\ 0 & a_2 & a_1 & a_0 \\ t_1a_2+t_2b_2 & t_1a_1+t_2b_1 & t_1a_0+t_2b_0 & 0 \\ 0 & t_1a_2+t_2b_2 & t_1a_1+t_2b_1 & t_1a_0+t_2b_0 \end{vmatrix}$$

$$+ s_2^2 \begin{vmatrix} b_2 & b_1 & b_0 & 0 \\ 0 & b_2 & b_1 & b_0 \\ t_1a_2+t_2b_2 & t_1a_1+t_2b_1 & t_1a_0+t_2b_0 & 0 \\ 0 & t_1a_2+t_2b_2 & t_1a_1+t_2b_1 & t_1a_0+t_2b_0 \end{vmatrix}.$$

We write it by

$$R(s_1f + s_2g, t_1f + t_2g) = s_1^2|A| + s_1s_2|B| + s_1s_2|C| + s_2^2|D|.$$

Then the first determinant $|A\ |$ can be decomposed into 4 smaller parts

$$|A\ | \quad = \begin{vmatrix} a_2 & a_1 & a_0 & 0 \\ 0 & a_2 & a_1 & a_0 \\ t_1a_2 & t_1a_1 & t_1a_0 & 0 \\ 0 & t_1a_2 & t_1a_1 & t_1a_0 \end{vmatrix} + \begin{vmatrix} a_2 & a_1 & a_0 & 0 \\ 0 & a_2 & a_1 & a_0 \\ t_1a_2 & t_1a_1 & t_1a_0 & 0 \\ 0 & t_2b_2 & t_2b_1 & t_2b_0 \end{vmatrix}$$

$$+ \begin{vmatrix} a_2 & a_1 & a_0 & 0 \\ 0 & a_2 & a_1 & a_0 \\ t_2b_2 & t_2b_1 & t_2b_0 & 0 \\ 0 & t_1a_2 & t_1a_1 & t_1a_0 \end{vmatrix} + \begin{vmatrix} a_2 & a_1 & a_0 & 0 \\ 0 & a_2 & a_1 & a_0 \\ t_2b_2 & t_2b_1 & t_2b_0 & 0 \\ 0 & t_2b_2 & t_2b_1 & t_2b_0 \end{vmatrix},$$

where the first three determinants are zero, hence

$$|A| = \begin{vmatrix} a_2 & a_1 & a_0 & 0 \\ 0 & a_2 & a_1 & a_0 \\ t_2b_2 & t_2b_1 & t_2b_0 & 0 \\ 0 & t_2b_2 & t_2b_1 & t_2b_0 \end{vmatrix} = t_2^2 \cdot R(f, g).$$

Similarly $|B|$ is decomposed into 4 parts where three of them are 0, so

$$|B| = \begin{vmatrix} a_2 & a_1 & a_0 & 0 \\ 0 & b_2 & b_1 & b_0 \\ t_2b_2 & t_2b_1 & t_2b_0 & 0 \\ 0 & t_1a_2 & t_1a_1 & t_1a_0 \end{vmatrix} = t_1t_2 \begin{vmatrix} a_2 & a_1 & a_0 & 0 \\ 0 & b_2 & b_1 & b_0 \\ b_2 & b_1 & b_0 & 0 \\ 0 & a_2 & a_1 & a_0 \end{vmatrix} = -t_1t_2R(f,g).$$

Analogous consideration for $|C|$ and $|D|$ gives rise to

$$\begin{aligned} R(s_1f+s_2g, t_1f+t_2g) &= s_1^2|A| + s_1s_2|B| + s_1s_2|C| + s_2^2|D| \\ &= (s_1^2t_2^2 - s_1s_2t_1t_2 - s_1s_2t_1t_2 + s_2^2t_1^2)\ R(f,g) \\ &= (s_1t_2 - s_2t_1)^2R(f,g) = \begin{vmatrix} s_1 & t_1 \\ s_2 & t_2 \end{vmatrix}^2 R(f,g). \end{aligned}$$

If $s_1 = s_2 = t_1 = 1$ and $t_2 = 0$ then $R(f+g,f) = (-1)^nR(f,g)$, this is Theorem 4.2. We also have the following theorem

Theorem 4.3. *Let $f(x), g(x) \in K[x]$ be of degree n. Then,*

$$R(s_1f+s_2g, t_1f+t_2g) = \begin{vmatrix} s_1 & t_1 \\ s_2 & t_2 \end{vmatrix}^n R(f,g), \ \textit{for } s_i, t_i \in K$$

Proof. Let $f(x) = \sum_{i=0}^n a_ix^i$ and $g(x) = \sum_{i=0}^n b_ix^i$. Then
$R(s_1f+s_2g, t_1f+t_2g)$

$$= \begin{vmatrix} s_1a_n+s_2b_n & \cdots & s_1a_1+s_2b_1 & s_1a_0+s_2b_0 & 0\ \cdots \\ 0 & 0 & s_1a_n+s_2b_n & \ddots & s_1a_1+s_2b_1\ \cdots \\ t_1a_n+t_2b_n & \cdots & t_1a_1+t_2b_1 & t_1a_0+t_2b_0 & 0\ \cdots \\ 0 & 0 & t_1a_n+t_2b_n & \ddots & t_1a_1+t_2b_1\ \cdots \end{vmatrix}.$$

This determinant can be decomposed into the sum of 2^{2n} determinants, but among them there are only 2^n nonzero determinants so that we can write
$R(s_1f+s_2g, t_1f+t_2g)$

$$= \begin{vmatrix} s_1a_n & \cdots & s_1a_1 & s_1a_0 & 0\ .. \\ 0 & 0 & s_1a_n & \ddots & s_1a_1\ .. \\ t_2b_n & \cdots & t_2b_1 & t_2b_0 & 0\ .. \\ 0 & 0 & t_2b_n & \ddots & t_2b_1\ .. \end{vmatrix} + \begin{vmatrix} s_1a_n & \cdots & s_1a_1 & s_1a_0 & 0\ .. \\ 0 & 0 & s_2b_n & \ddots & s_2b_1\ .. \\ t_2b_n & \cdots & t_2b_1 & t_2b_0 & 0\ .. \\ 0 & 0 & t_1a_n & \ddots & t_1a_1\ .. \end{vmatrix}$$

$+\cdots+$

$$\begin{vmatrix} s_1a_n & \cdots & s_1a_1 & s_1a_0 & 0 & .. \\ 0 & 0 & s_2b_n & \ddots & s_2b_1 & .. \\ t_2b_n & \cdots & t_2b_1 & t_2b_0 & 0 & .. \\ 0 & 0 & t_1a_n & \ddots & t_1a_1 & .. \end{vmatrix} + \begin{vmatrix} s_2b_n & \cdots & s_2b_1 & s_2b_0 & 0 & .. \\ 0 & 0 & s_2b_n & \ddots & s_2b_1 & .. \\ t_1a_n & \cdots & t_1a_1 & t_1a_0 & 0 & .. \\ 0 & 0 & t_1a_n & \ddots & t_1a_1 & .. \end{vmatrix}$$

$$= \Big((s_1t_2)^n + (s_1t_2)^{n-1}(-s_2t_1) + \cdots + s_1t_2(-s_2t_1)^{n-1} + (-s_2t_1)^n\Big)R(f,g)$$

$$= (s_1t_2 - s_2t_1)^n \; R(f,g) = \begin{vmatrix} s_1 & t_1 \\ s_2 & t_2 \end{vmatrix}^n R(f,g). \qquad \square$$

We generalize this to linear combination of three polynomials.

THEOREM 4.4. *If f_i $(1 \le i < n)$ are linear polynomials then* $R(\sum_{i=1}^n s_if_i, \sum_{i=1}^n t_if_i) = \sum_{1\le i<j\le n} D_{ij}R(f_i,f_j)$ *for* $D_{ij} = \begin{vmatrix} s_i & t_i \\ s_j & t_j \end{vmatrix}$.

Proof. If $f_1(x) = a_1x + a_0,\ f_2(x) = b_1x + b_0,\ f_3(x) = c_1x + c_0$ then

$$R(s_1f_1 + s_2f_2 + s_3f_3, t_1f_1 + t_2f_2 + t_3f_3)$$
$$= \begin{vmatrix} s_1 & t_1 \\ s_2 & t_2 \end{vmatrix} R(f_1,f_2) + \begin{vmatrix} s_1 & t_1 \\ s_3 & t_3 \end{vmatrix} R(f_1,f_3) + \begin{vmatrix} s_2 & t_2 \\ s_3 & t_3 \end{vmatrix} R(f_2,f_3).$$

$\square$

Now if $f_1 = \sum_{i=0}^2 a_ix^i,\ f_2 = \sum_{i=0}^2 b_ix^i$ and $f_3 = \sum_{i=0}^2 c_ix^i$ are quadratic then we have

$$R(s_1f_1 + s_2f_2 + s_3f_3, t_1f_1 + t_2f_2 + t_3f_3)$$
$$= \begin{vmatrix} s_1a_2+s_2b_2+s_3c_2 & s_1a_1+s_2b_1+s_3c_1 & s_1a_0+s_2b_0+s_3c_0 & \cdots \\ 0 & s_1a_2+s_2b_2+s_3c_2 & s_1a_1+s_2b_1+s_3c_1 & \cdots \\ t_1a_2+t_2b_2+t_3c_2 & t_1a_1+t_2b_1+t_3c_1 & t_1a_0+t_2b_0+t_3c_0 & \cdots \\ 0 & t_1a_2+t_2b_2+t_3c_2 & t_1a_1+t_2b_1+t_3c_1 & \cdots \end{vmatrix}$$

Let $\mathbf{A} = [a_2\ a_1\ a_0]$, $\mathbf{B} = [b_2\ b_1\ b_0]$ and $\mathbf{C} = [c_2\ c_1\ c_0]$. Then

$$R(s_1f_1 + s_2f_2 + s_3f_3,\ t_1f_1 + t_2f_2 + t_3f_3)$$

$$= s_1^2t_2^2 \begin{vmatrix} \mathbf{A} & 0 \\ 0 & \mathbf{A} \\ \mathbf{B} & 0 \\ 0 & \mathbf{B} \end{vmatrix} - s_1s_2t_1t_2 \begin{vmatrix} \mathbf{A} & 0 \\ 0 & \mathbf{A} \\ \mathbf{B} & 0 \\ 0 & \mathbf{B} \end{vmatrix} - s_1s_3t_1t_2 \begin{vmatrix} \mathbf{A} & 0 \\ 0 & \mathbf{A} \\ \mathbf{B} & 0 \\ 0 & \mathbf{C} \end{vmatrix} + s_1^2t_2t_3 \begin{vmatrix} \mathbf{A} & 0 \\ 0 & \mathbf{A} \\ \mathbf{B} & 0 \\ 0 & \mathbf{C} \end{vmatrix}$$
$$-s_1s_2t_2t_3 \begin{vmatrix} \mathbf{A} & 0 \\ 0 & \mathbf{C} \\ \mathbf{B} & 0 \\ 0 & \mathbf{B} \end{vmatrix} + s_1s_3t_2^2 \begin{vmatrix} \mathbf{A} & 0 \\ 0 & \mathbf{C} \\ \mathbf{B} & 0 \\ 0 & \mathbf{B} \end{vmatrix} + s_1^2t_2t_3 \begin{vmatrix} \mathbf{A} & 0 \\ 0 & \mathbf{A} \\ \mathbf{C} & 0 \\ 0 & \mathbf{B} \end{vmatrix} - s_1s_2t_1t_3 \begin{vmatrix} \mathbf{A} & 0 \\ 0 & \mathbf{A} \\ \mathbf{C} & 0 \\ 0 & \mathbf{B} \end{vmatrix}$$

$$-s_1s_3t_1t_3\begin{vmatrix}\mathbf{A}&0\\0&\mathbf{A}\\\mathbf{C}&0\\0&\mathbf{C}\end{vmatrix}+s_1^2t_3^2\begin{vmatrix}\mathbf{A}&0\\0&\mathbf{A}\\\mathbf{C}&0\\0&\mathbf{C}\end{vmatrix}+s_1s_2t_3^2\begin{vmatrix}\mathbf{A}&0\\0&\mathbf{B}\\\mathbf{C}&0\\0&\mathbf{C}\end{vmatrix}-s_1s_3t_2t_3\begin{vmatrix}\mathbf{A}&0\\0&\mathbf{B}\\\mathbf{C}&0\\0&\mathbf{C}\end{vmatrix}$$

$$-s_1s_2t_1t_2\begin{vmatrix}\mathbf{A}&0\\0&\mathbf{A}\\\mathbf{B}&0\\0&\mathbf{B}\end{vmatrix}+s_2^2t_1^2\begin{vmatrix}\mathbf{A}&0\\0&\mathbf{A}\\\mathbf{B}&0\\0&\mathbf{B}\end{vmatrix}+s_2s_3t_1^2\begin{vmatrix}\mathbf{A}&0\\0&\mathbf{A}\\\mathbf{B}&0\\0&\mathbf{C}\end{vmatrix}-s_1s_2t_1t_3\begin{vmatrix}\mathbf{A}&0\\0&\mathbf{A}\\\mathbf{B}&0\\0&\mathbf{C}\end{vmatrix}$$

$$+s_2^2t_1t_3\begin{vmatrix}\mathbf{A}&0\\0&\mathbf{C}\\\mathbf{B}&0\\0&\mathbf{B}\end{vmatrix}-s_2s_3t_1t_2\begin{vmatrix}\mathbf{A}&0\\0&\mathbf{C}\\\mathbf{B}&0\\0&\mathbf{B}\end{vmatrix}-s_1s_2t_2t_3\begin{vmatrix}\mathbf{C}&0\\0&\mathbf{A}\\\mathbf{B}&0\\0&\mathbf{B}\end{vmatrix}+s_2^2t_1t_3\begin{vmatrix}\mathbf{C}&0\\0&\mathbf{A}\\\mathbf{B}&0\\0&\mathbf{B}\end{vmatrix}$$

$$-s_2s_3t_1t_3\begin{vmatrix}\mathbf{B}&0\\0&\mathbf{A}\\\mathbf{C}&0\\0&\mathbf{C}\end{vmatrix}+s_1s_2t_3^2\begin{vmatrix}\mathbf{B}&0\\0&\mathbf{A}\\\mathbf{C}&0\\0&\mathbf{C}\end{vmatrix}+s_2^2t_3^2\begin{vmatrix}\mathbf{B}&0\\0&\mathbf{B}\\\mathbf{C}&0\\0&\mathbf{C}\end{vmatrix}-s_2s_3t_2t_3\begin{vmatrix}\mathbf{B}&0\\0&\mathbf{B}\\\mathbf{C}&0\\0&\mathbf{C}\end{vmatrix}$$

$$-s_1s_3t_1t_2\begin{vmatrix}\mathbf{A}&0\\0&\mathbf{A}\\\mathbf{C}&0\\0&\mathbf{B}\end{vmatrix}+s_2s_3t_1^2\begin{vmatrix}\mathbf{A}&0\\0&\mathbf{A}\\\mathbf{C}&0\\0&\mathbf{B}\end{vmatrix}+s_3^2t_1^2\begin{vmatrix}\mathbf{A}&0\\0&\mathbf{A}\\\mathbf{C}&0\\0&\mathbf{C}\end{vmatrix}-s_1s_3t_1t_3\begin{vmatrix}\mathbf{A}&0\\0&\mathbf{A}\\\mathbf{C}&0\\0&\mathbf{C}\end{vmatrix}$$

$$-s_2s_3t_1t_3\begin{vmatrix}\mathbf{A}&0\\0&\mathbf{B}\\\mathbf{C}&0\\0&\mathbf{C}\end{vmatrix}+s_3^2t_1t_2\begin{vmatrix}\mathbf{A}&0\\0&\mathbf{B}\\\mathbf{C}&0\\0&\mathbf{C}\end{vmatrix}+s_1s_3t_2^2\begin{vmatrix}\mathbf{C}&0\\0&\mathbf{A}\\\mathbf{B}&0\\0&\mathbf{B}\end{vmatrix}-s_2s_3t_1t_2\begin{vmatrix}\mathbf{C}&0\\0&\mathbf{A}\\\mathbf{B}&0\\0&\mathbf{B}\end{vmatrix}$$

$$+s_3^2t_1t_2\begin{vmatrix}\mathbf{B}&0\\0&\mathbf{A}\\\mathbf{C}&0\\0&\mathbf{C}\end{vmatrix}-s_1s_3t_2t_3\begin{vmatrix}\mathbf{B}&0\\0&\mathbf{A}\\\mathbf{C}&0\\0&\mathbf{C}\end{vmatrix}-s_2s_3t_2t_3\begin{vmatrix}\mathbf{B}&0\\0&\mathbf{B}\\\mathbf{C}&0\\0&\mathbf{C}\end{vmatrix}+s_3^2t_2^2\begin{vmatrix}\mathbf{B}&0\\0&\mathbf{B}\\\mathbf{C}&0\\0&\mathbf{C}\end{vmatrix}.$$

Thus

$$R(s_1f_1+s_2f_2+s_3f_3, t_1f_1+t_2f_2+t_3f_3)$$

$$= D_{12}^2R(f_1,f_2)+D_{13}^2R(f_1,f_3)+D_{23}^2R(f_2,f_3)$$

$$+(-s_1s_3t_1t_2+s_1^2t_2t_3+s_2s_3t_1^2-s_1s_2t_1t_3)\left(\begin{vmatrix}\mathbf{A}&0\\0&\mathbf{A}\\\mathbf{B}&0\\0&\mathbf{C}\end{vmatrix}+\begin{vmatrix}\mathbf{A}&0\\0&\mathbf{A}\\\mathbf{C}&0\\0&\mathbf{B}\end{vmatrix}\right)$$

$$+(-s_1s_2t_2t_3+s_1s_3t_2^2+s_2^2t_1t_3-s_2s_3t_1t_2)\left(\begin{vmatrix}\mathbf{A}&0\\0&\mathbf{C}\\\mathbf{B}&0\\0&\mathbf{B}\end{vmatrix}+\begin{vmatrix}\mathbf{C}&0\\0&\mathbf{A}\\\mathbf{B}&0\\0&\mathbf{B}\end{vmatrix}\right)$$

$$+\,(s_1s_2t_3^2 - s_1s_3t_2t_3 - s_2s_3t_1t_3 + s_3^2t_1t_2)\left(\begin{vmatrix} \mathbf{A} & 0 \\ 0 & \mathbf{B} \\ \mathbf{C} & 0 \\ 0 & \mathbf{C} \end{vmatrix} + \begin{vmatrix} \mathbf{B} & 0 \\ 0 & \mathbf{A} \\ \mathbf{C} & 0 \\ 0 & \mathbf{C} \end{vmatrix}\right)$$

$$= D_{12}^2R(f_1,f_2) + D_{13}^2R(f_1,f_3) + D_{23}^2R(f_2,f_3)$$

$$+\,D_{12}D_{13}\left(\begin{vmatrix} \mathbf{A} & 0 \\ 0 & \mathbf{A} \\ \mathbf{B} & 0 \\ 0 & \mathbf{C} \end{vmatrix} + \begin{vmatrix} \mathbf{A} & 0 \\ 0 & \mathbf{A} \\ \mathbf{C} & 0 \\ 0 & \mathbf{B} \end{vmatrix}\right) - D_{12}D_{23}\left(\begin{vmatrix} \mathbf{A} & 0 \\ 0 & \mathbf{C} \\ \mathbf{B} & 0 \\ 0 & \mathbf{B} \end{vmatrix} + \begin{vmatrix} \mathbf{C} & 0 \\ 0 & \mathbf{A} \\ \mathbf{B} & 0 \\ 0 & \mathbf{B} \end{vmatrix}\right)$$

$$+\,D_{13}D_{23}\left(\begin{vmatrix} \mathbf{A} & 0 \\ 0 & \mathbf{B} \\ \mathbf{C} & 0 \\ 0 & \mathbf{C} \end{vmatrix} + \begin{vmatrix} \mathbf{B} & 0 \\ 0 & \mathbf{A} \\ \mathbf{C} & 0 \\ 0 & \mathbf{C} \end{vmatrix}\right)$$

where $D_{12} = \begin{vmatrix} s_1 & t_1 \\ s_2 & t_2 \end{vmatrix}$, $D_{13} = \begin{vmatrix} s_1 & t_1 \\ s_3 & t_3 \end{vmatrix}$ and $D_{23} = \begin{vmatrix} s_2 & t_2 \\ s_3 & t_3 \end{vmatrix}$.

Therefore it gives rise to the following theorem.

THEOREM 4.5. *If* $f_1 = \sum_{i=0}^{2} a_i x^i,\ f_2 = \sum_{i=0}^{2} b_i x^i,\ f_3 = \sum_{i=0}^{2} c_i x^i$ *then*

$$R(\sum_{j=1}^{3} s_j f_j, \sum_{j=1}^{3} t_j f_j) = D_{12}^2R(f_1,f_2) + D_{13}^2R(f_1,f_3) + D_{23}^2R(f_2,f_3)$$

$$+2\left(D_{12}D_{13}\begin{vmatrix} \mathbf{A} & 0 \\ 0 & \mathbf{A} \\ \mathbf{B} & 0 \\ 0 & \mathbf{C} \end{vmatrix} - D_{12}D_{23}\begin{vmatrix} \mathbf{A} & 0 \\ 0 & \mathbf{C} \\ \mathbf{B} & 0 \\ 0 & \mathbf{B} \end{vmatrix} + D_{13}D_{23}\begin{vmatrix} \mathbf{A} & 0 \\ 0 & \mathbf{B} \\ \mathbf{C} & 0 \\ 0 & \mathbf{C} \end{vmatrix}\right)$$

where $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ *and* D_{ij} *are as above. Furthermore*

$$R(\sum_{j=1}^{3} s_j f_j, \sum_{j=1}^{3} t_j f_j) = D_{12}^2R(f_1,f_2) + D_{13}^2R(f_1,f_3) + D_{23}^2R(f_2,f_3)$$

$$+\,2D_{12}D_{13}\left((AB)_{20}(AC)_{20} - (AB)_{21}(AC)_{10}\right)$$

$$+\,2D_{12}D_{23}\left((AB)_{20}(BC)_{20} - (AB)_{21}(BC)_{10}\right)$$

$$+\,2D_{13}D_{23}\left((AC)_{20}(BC)_{20} - (AC)_{21}(BC)_{10}\right),$$

where $(AB)_{ij} = \begin{vmatrix} a_i & a_j \\ b_i & b_j \end{vmatrix}$, $(AC)_{ij} = \begin{vmatrix} a_i & a_j \\ c_i & c_j \end{vmatrix}$ *and* $(BC)_{ij} = \begin{vmatrix} b_i & b_j \\ c_i & c_j \end{vmatrix}$.

Proof. It is easy to see $\begin{vmatrix} \mathbf{A} & 0 \\ 0 & \mathbf{A} \\ \mathbf{B} & 0 \\ 0 & \mathbf{C} \end{vmatrix} = \begin{vmatrix} \mathbf{A} & 0 \\ 0 & \mathbf{A} \\ \mathbf{C} & 0 \\ 0 & \mathbf{B} \end{vmatrix}$. Since $\begin{vmatrix} a & b & c & 0 \\ d & e & f & 0 \\ 0 & g & h & i \\ 0 & j & k & l \end{vmatrix} =$ $\begin{vmatrix} a & b \\ d & e \end{vmatrix} \begin{vmatrix} h & i \\ k & l \end{vmatrix} - \begin{vmatrix} a & c \\ d & f \end{vmatrix} \begin{vmatrix} g & i \\ j & l \end{vmatrix}$, we have

$$\begin{aligned} \begin{vmatrix} \mathbf{A} & 0 \\ 0 & \mathbf{A} \\ \mathbf{B} & 0 \\ 0 & \mathbf{C} \end{vmatrix} = -\begin{vmatrix} \mathbf{A} & 0 \\ \mathbf{B} & 0 \\ 0 & \mathbf{A} \\ 0 & \mathbf{C} \end{vmatrix} &= -\begin{vmatrix} a_2 & a_1 \\ b_2 & b_1 \end{vmatrix} \begin{vmatrix} a_1 & a_0 \\ c_1 & c_0 \end{vmatrix} + \begin{vmatrix} a_2 & a_0 \\ b_2 & b_0 \end{vmatrix} \begin{vmatrix} a_2 & a_0 \\ c_2 & c_0 \end{vmatrix} \\ &= (AB)_{20}(AC)_{20} - (AB)_{21}(AC)_{10}, \end{aligned}$$

$$\begin{aligned} \begin{vmatrix} \mathbf{A} & 0 \\ 0 & \mathbf{C} \\ \mathbf{B} & 0 \\ 0 & \mathbf{B} \end{vmatrix} = -\begin{vmatrix} \mathbf{A} & 0 \\ \mathbf{B} & 0 \\ 0 & \mathbf{C} \\ 0 & \mathbf{B} \end{vmatrix} &= -\begin{vmatrix} a_2 & a_1 \\ b_2 & b_1 \end{vmatrix} \begin{vmatrix} c_1 & c_0 \\ b_1 & b_0 \end{vmatrix} + \begin{vmatrix} a_2 & a_0 \\ b_2 & b_0 \end{vmatrix} \begin{vmatrix} c_2 & c_0 \\ b_2 & b_0 \end{vmatrix} \\ &= (AB)_{20}(CB)_{20} - (AB)_{21}(CB)_{10}, \end{aligned}$$

$$\begin{aligned} \begin{vmatrix} \mathbf{A} & 0 \\ 0 & \mathbf{B} \\ \mathbf{C} & 0 \\ 0 & \mathbf{C} \end{vmatrix} = -\begin{vmatrix} \mathbf{A} & 0 \\ \mathbf{C} & 0 \\ 0 & \mathbf{B} \\ 0 & \mathbf{C} \end{vmatrix} &= -\begin{vmatrix} a_2 & a_1 \\ c_2 & c_1 \end{vmatrix} \begin{vmatrix} b_1 & b_0 \\ c_1 & c_0 \end{vmatrix} + \begin{vmatrix} a_2 & a_0 \\ c_2 & c_0 \end{vmatrix} \begin{vmatrix} b_2 & b_0 \\ c_2 & c_0 \end{vmatrix} \\ &= (AC)_{20}(BC)_{20} - (AC)_{21}(BC)_{10}, \end{aligned}$$

so it follows the conclusion. □

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