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A COMMON FIXED POINT THEOREM ON FUZZY 2-METRIC SPACES

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Abstract. In this paper, we prove a common fixed point theorem for four mappings on fuzzy 2-metric spaces. Our result is an extension of results of S. H. Cho [2] to fuzzy 2-metric spaces. Also, it is a generalization of a result of S. Sharma [11].

1. Introduction

The concept of fuzzy sets was introduced by L. A. Zadeh [13] in 1965. To use this concept in topology and analysis, many authors have extensively developed the theory of fuzzy sets and applications. With the concept of fuzzy sets, the fuzzy metric space was introduced by I. Kramosil and J. Michalek [8] in 1975. M. Grabiec [5] proved the contraction principle in fuzzy metric spaces in 1988. Moreover, A. George and P. Veeramani [4] modified the notion of fuzzy metric spaces with the help of t-norms in 1994. Gähler [3] investigated 2-metric spaces in a series of his papers. Sharma, Sharma and Iseki [12] investigated, for the first time, contraction type mappings in 2-metric spaces. Many authors have studied common fixed point theorems in fuzzy metric spaces. Some of interesting papers are Y. J. Cho [1], George and Veeramani [4], Grabiec [5], Kramosil and Michalek [8] and S. Sharma [11].

S. H. Cho [2] proved a common fixed point theorem for four mappings in fuzzy metric spaces and S. Sharma [11] proved a common fixed point theorem for three mappings in fuzzy 2-metric spaces. In this paper we prove a common fixed point theorem for four mappings in fuzzy 2-metric spaces. Our theorem is an extension of results of S. H. Cho [2] to fuzzy 2-metric spaces. And also, it is a generalization of result of and S. Sharma [11].

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2. Preliminaries

Now we begin with some definitions:

Definition 2.1. (Schweizer and Sklar [10]) A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is called a continuous t-norm if $([0,1],*)$ is an Abelian topological monoid with unit 1 such that $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ for all $a,b,c,d \in [0,1]$.

Examples of t-norms are $a * b = ab$ and $a * b = \min\{a,b\}$. We use a prefix notation $\triangle(x,y)$ instead of the infix notation $x * y$ for $x,y \in [0,1]$. A t-norm $\triangle$ is said to be H-type [6] if the family of $\{\triangle^m(t)\}_{m=1}^{\infty}$ is equicontinuous at $t=1$, where $\triangle^1(t) = \triangle(t,t)$, $\triangle^m(t) = \triangle(t, \triangle^{m-1}(t))$, $m = 1,2,\cdots$, and $t \in [0,1]$.

Note that a t-norm $\triangle$ is of H-type if and only if for any $\epsilon \in (0,1)$, there exists $\delta \in (0,1)$ such that $x > 1-\delta$ implies $\triangle^n(x) > 1-\epsilon$ for all $n \geq 1$. It ease to see that the minimum t-norm is of H-type. It is known that the only t-norm $\triangle$ satisfying $\triangle(s,s) \geq s$ for all $s \in [0,1]$ is the minimum t-norm (see[7]). So, the t-norm $\triangle$ satisfying $\triangle(s,s) \geq s$ for all $s \in [0,1]$ is of H-type.

Definition 2.2. (I. Kramosil, J. Michalek [8]) The 3-tuple $(X,M,\triangle)$ is called a fuzzy metric space if X is an arbitrary set, $\triangle$ is a continuous t-norm of H-type and M is a fuzzy set in $X^2 \times [0,\infty)$ satisfying the following conditions:

(1) $M(x,y,0) = 0$,
(2) $M(x,y,t) = 1$ for all $t > 0$ if and only if $x = y$,
(3) $M(x,y,t) = M(y,x,t)$,
(4) $M(x,z,t+s) \geq \triangle(M(x,y,t), M(y,z,s))$,
(5) $M(x,y,\cdot) : [0,1) \to (0,1]$ is left continuous for all $x,y,z \in X$ and $t,s > 0$.

Note that $M(x,y,t)$ can be thought of as the degree of nearness between x and y with respect to t.

Example 2.3. Let (X,d) be a metric space. Define $a * b = ab$ or $a * b = \min\{a,b\}$ and for all $x,y \in X$ and $t > 0$,

$$M(x,y,t) = \frac{t}{t + d(x,y)}.$$

Then $(X,M,*)$ is a fuzzy metric space. We call this fuzzy metric M induced by the metric d the standard fuzzy metric.

Definition 2.4. (Gähler [3]) Let X be a nonempty set. A real valued function d on $X \times X \times X$ is said to be a 2-metric on X if

(1) given distinct elements x, y of X, there exists an element $z \in X$ such that $d(x, y, z) \neq 0$,
(2) $d(x, y, z) = 0$ when at least two of $x, y, z \in X$ are equal,
(3) $d(x, y, z) = d(x, z, y) = d(y, z, x)$ for all $x, y, z \in X$,
(4) $d(x, y, z) \leq d(x, y, w) + d(x, w, z) + d(w, y, z)$ for all $x, y, z, w \in X$.
The pair (X, d) is called a 2-metric space.

Example 2.5. Let $X = \mathbb{R}^3$ and let $d(x, y, z) :=$ the area of the triangle spanned by x, y and z which may be given explicitly by the formula,
$d(x, y, z) = |x_1(y_2z_3 - y_3z_2) - x_2(y_1z_3 - y_3z_1) + x_3(y_1z_2 - y_2z_1)|$, where $x = (x_1, x_2, x_3), y = (y_1, y_2, y_3), z = (z_1, z_2, z_3)$.
Then (X, d) is a 2-metric space.

Definition 2.6. (S. Sharma [11]) The 3-tuple $(X, M, \triangle)$ is called a fuzzy 2-metric space if X is an arbitrary set, $\triangle$ is a continuous t-norm of H-type and M is a fuzzy set in $X^3 \times [0, \infty)$ satisfying the following conditions : for all $x, y, z, u \in X$ and $t_1, t_2, t_3 > 0$,

(1) $M(x, y, z, 0) = 0$,
(2) $M(x, y, z, t) = 1$ for all $t > 0$ if and only if at least two of the three points are equal,
(3) $M(x, y, z, t) = M(x, z, y, t) = M(y, z, x, t)$ for all $t > 0$,
(Symmetry about first three variables)
(4) $M(x, y, z, t_1 + t_2 + t_3)$
$\geq \triangle(M(x, y, u, t_1), M(x, u, z, t_2), M(u, y, z, t_3))$,
(This corresponds to tetrahedron inequality in 2-metric space. The function value $M(x, y, z, t)$ may be interpreted as the probability that the area of triangle is less than t.)
(5) $M(x, y, z, \cdot) : [0, 1) \to [0, 1]$ is left continuous .

Example 2.7. Let (X, d) be a 2-metric space and denote $\triangle(a, b) = ab$ for all $a, b \in [0, 1]$.

For each $h, m, n \in \mathbb{R}^+$ and $t > 0$, define $M(x, y, z, t) = \frac{ht^n}{ht^n + md(x,y,z)}$.
Then $(X, M, \triangle)$ is an fuzzy 2-metric space.

Definition 2.8. Let $(X, M, \triangle)$ be a fuzzy 2-metric space.

(1) A sequence $\{x_n\}$ in fuzzy 2-metric space X is said to be convergent to a point $x \in X$ (denoted by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$) if for any $\lambda \in (0, 1)$ and $t > 0$, there exists $n_0 \in N$ such that for all $n \geq n_0$

and $a \in X$, $M(x_n, x, a, t) > 1 - \lambda$. That is, $\lim_{n\to\infty} M(x_n, x, a, t) = 1$ for all $a \in X$ and $t > 0$.

(2) A sequence $\{x_n\}$ in fuzzy 2-metric space X is called a Cauchy sequence, if for any $\lambda \in (0, 1)$ and $t > 0$, there exists $n_0 \in N$ such that, for all $m, n \geq n_0$ and $a \in X$, $M(x_n, x_m, a, t) > 1 - \lambda$.

(3) A fuzzy 2-metric space in which every Cauchy sequence is convergent is said to be complete.

Definition 2.9. Self mappings A and B of a fuzzy 2-metric space $(X, M, \triangle)$ is said to be compatible, if $\lim_{n\to\infty} M(ABx_n, BAx_n, a, t) = 1$ for all $a \in X$ and $t > 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = z$ for some $z \in X$.

3. Main result

Lemma 3.1. *$M(x, y, z, \cdot)$ is non-decreasing for all $x, y, z \in X$.*

Proof. Let $s, t > 0$ be any points such that $t > s$. Then $t = s + \frac{t-s}{2} + \frac{t-s}{2}$. Hence we have

$$\begin{aligned} M(x, y, z, t) &= M(x, y, z, s + \frac{t-s}{2} + \frac{t-s}{2}) \\ &\geq \triangle(M(x, y, z, s), M(x, z, z, \frac{t-s}{2}), M(y, y, z, \frac{t-s}{2}) \\ &= M(x, y, z, s). \end{aligned}$$

Thus, $M(x, y, z, t) > M(x, y, z, s)$, □

From now on, let $(X, M, \triangle)$ be a fuzzy 2-metric space with the following condition : $\lim_{t\to\infty} M(x, y, z, t) = 1$ for all $x, y, z \in X$.

Lemma 3.2. *Let $(X, M, \triangle)$ be a fuzzy 2-metric space. If there exists $q \in (0, 1)$ such that $M(x, y, z, qt + 0) \geq M(x, y, z, t)$ for all $x, y, z \in X$ with $z \neq x$, $z \neq y$ and $t > 0$, then $x = y$.*

Proof. Since $M(x, y, z, t) \geq M(x, y, z, qt + 0) \geq M(x, y, z, t)$ for all $t > 0$, $M(x, y, z, \cdot)$ is constant. Since $\lim_{t\to\infty} M(x, y, z, t) = 1$, $M(x, y, z, t) = 1$ for all $t > 0$. Hence, $x = y$ because $x \neq z$ and $y \neq z$. □

Lemma 3.3. *Let $(X, M, \triangle)$ be a fuzzy 2-metric space, and let*

$$\lim_{n\to\infty} x_n = x, \lim_{n\to\infty} y_n = y, \lim_{n\to\infty} u_n = u \text{ and } \lim_{n\to\infty} v_n = v.$$

Then the followings are satisfied .
(1) $\liminf M(x_n, y_n, a, t) \geq M(x, y, a, t)$ *for all* $a \in X$ *and* $t \geq 0$.
(2) $M(x, y, a, t+0) \geq \limsup M(x_n, y_n, a, t)$ *for all* $a \in X$ *and* $t > 0$.

Proof. (1) For all $a \in X$ and $t > 0$, we have

$$\begin{aligned} M(x_n, y_n, a, t) &\geq \triangle(M(x_n, x, a, t), M(x_n, y_n, x, t), M(y_n, x, a, t)) \\ &\geq \triangle(M(x_n, x, a, t), M(x_n, x, y_n, t), M(y_n, y, a, t), M(x, y, a, t), \\ &\quad M(y_n, x, y, t)) \end{aligned}$$

which implies

$$\liminf M(x_n, y_n, a, t) \geq \triangle(1, 1, 1, M(x, y, a, t), 1) = M(x, y, a, t)$$

for all $a \in X$ and $t > 0$.
(2) Let $\epsilon > 0$ be given. For all $a \in X$ and $t > 0$, we have

$$\begin{aligned} M(x, y, a, t+2\epsilon) &\geq \triangle(M(x, x_n, a, \frac{\epsilon}{2}), M(x_n, y, a, t+\epsilon), M(x, y, x_n, \frac{\epsilon}{2})) \\ &\geq \triangle(M(x_n, x, a, \frac{\epsilon}{2}), M(x_n, x, y, \frac{\epsilon}{2}), M(x_n, y_n, a, t), M(x_n, y, y_n, \frac{\epsilon}{2}), \\ &\quad M(y_n, y, a, \frac{\epsilon}{2})) \end{aligned}$$

which implies $M(x, y, a, t+2\epsilon) \geq \limsup M(x_n, y_n, a, t)$. Letting $\epsilon \to 0$ in the above inequality, we have $M(x, y, a, t+0) \geq \limsup M(x_n, y_n, a, t)$. □

Note that for all $a \in X$ and $t > 0$, in general the inequality

$$M(x, y, a, t) \geq \limsup M(x_n, y_n, a, t)$$

is not true, because $M(x, y, z, \cdot)$ is left continuous (in general, not right continuous).

LEMMA 3.4. *Let* $(X, M, \triangle)$ *be a fuzzy 2-metric space and let* A *and* B *be continuous self mappings of* X *and* $[A, B]$ *be compatible. Let* x_n *be a sequence in* X *such that* $Ax_n \to z$ *and* $Bx_n \to z$. *Then* $ABx_n \to Bz$.

Proof. Since A, B are continuous maps, $ABx_n \to Az$, $BAx_n \to Bz$ and so, $M(ABx_n, Az, a, \frac{t}{3}) \to 1$ and $M(BAx_n, Bz, a, \frac{t}{3}) \to 1$ for all $a \in X$ and $t > 0$.
Since the pair $[A, B]$ is compatible, $M(BAx_n, ABx_n, a, \frac{t}{3}) \to 1$ for all

$a \in X$ and $t > 0$. Thus,

$$\begin{aligned} &M(ABx_n, Bz, a, t) \\ &\quad \geq \triangle(M(ABx_n, Bz, BAx_n, \frac{t}{3}), M(ABx_n, BAx_n, a, \frac{t}{3}), \\ &\qquad M(BAx_n, Bz, a, \frac{t}{3})) \\ &\quad \geq \triangle(M(BAx_n, Bz, ABx_n, \frac{t}{3}), M(BAx_n, ABx_n, a, \frac{t}{3}), \\ &\qquad M(BAx_n, Bz, a, \frac{t}{3})) \\ &\quad \to 1 \end{aligned}$$

for all $a \in X$ and $t > 0$.
Hence, $ABx_n \to Bz$. □

Theorem 3.5. *Let $(X, M, \triangle)$ be a complete fuzzy 2-metric space with continuous t-norm $\triangle$ of H-type, and let S and T be continuous self mappings of X. Then S and T have a unique common fixed point in X if and only if there exist two self mappings A, B of X satisfying*

(1) $AX \subset TX,\ BX \subset SX$,
(2) *the pair $[A, S]$ and $[B, T]$ are compatible,*
(3) *there exists $q \in (0, 1)$ such that for every $x, y, a \in X$ and $t > 0$,*
$$\begin{aligned} &M(Ax, By, a, qt) \\ &\quad \geq \min\{M(Sx, Ty, a, t), M(Ax, Sx, a, t), M(By, Ty, a, t), \\ &\quad M(Ax, Ty, a, t)\}. \end{aligned} \tag{3.0}$$

Indeed, A, B, S and T have a unique common fixed point in X.

Proof. Suppose that S and T have a (unique) common fixed point, say $z \in X$.
Define $A : X \to X$ by $Ax = z$ for all $x \in X$, and $B : X \to X$ by $Bx = z$ for all $x \in X$.
Then one can see that (1)- (3) are satisfied.

Conversely, assume that there exist two self mappings A, B of X satisfying conditions (1)- (3). From condition (1) we can construct two sequences $\{x_n\}$ and $\{y_n\}$ of X such that $y_{2n-1} = Tx_{2n-1} = Ax_{2n-2}$ and $y_{2n} = Sx_{2n} = Bx_{2n-1}$ for $n = 1, 2, \cdots$. Putting $x = x_{2n}$ and $y = x_{2n+1}$

in (3.0), we have that for all $a \in X$ and $t > 0$

$$\begin{aligned} M(y_{2n+1}, y_{2n+2}, a, qt) &= M(Ax_{2n}, Bx_{2n+1}, a, qt) \\ &\geq \min\{M(Sx_{2n}, Tx_{2n+1}, a, t), M(Ax_{2n}, Sx_{2n}, a, t), \\ &\quad M(Bx_{2n+1}, Tx_{2n+1}, a, t), M(Ax_{2n}, Tx_{2n+1}, a, t)\} \\ &= \min\{M(y_{2n}, y_{2n+1}, a, qt), M(y_{2n+1}, y_{2n+2}, a, qt)\} \end{aligned}$$

which implies $M(y_{2n+1}, y_{2n+2}, a, qt) \geq M(y_{2n}, y_{2n+1}, a, t)$ by Lemma 3.1. Also, letting $x = x_{2n+2}$ and $y = x_{2n+1}$ in (3.0), we have that $M(y_{2n+2}, y_{2n+3}, a, qt) \geq M(y_{2n+1}, y_{2n+2}, a, t)$ for all $a \in X$ and $t > 0$. In general, we obtain that for all $a \in X, t > 0$ and $n = 1, 2, \cdots$ $M(y_n, y_{n+1}, a, qt) \geq M(y_{n-1}, y_n, a, t)$. Thus, for all $a \in X$, $t > 0$ and $n = 1, 2, \cdots$

$$M(y_n, y_{n+1}, a, t) \geq M(y_0, y_1, a, \tfrac{t}{q^n}). \tag{3.1}$$

We now show that $\{y_n\}$ is a Cauchy sequence in X.
Let $\epsilon \in (0, 1)$ be given. Since the t-norm $\triangle$ is of H-type, there exists $\lambda \in (0, 1)$ such that for all $m, n \in \mathbb{N}$ with $m > n$

$$\triangle^{2^{m-n}}(1-\lambda) > 1-\epsilon. \tag{3.2}$$

Since $\lim_{n\to\infty} M(y_0, y_1, a, \frac{t}{q^n}) = 1$, there exists $n_0 \in \mathbb{N}$ such that for all $a \in X$ and $t > 0$ with $M(y_0, y_1, a, \frac{t}{q^n}) > 1 - \lambda$ for all $n \geq n_0$. From (3.1) we have that for all $a \in X$ and $t > 0$, $M(y_n, y_{n+1}, a, t) > 1 - \lambda$ for all $n \geq n_0$.
Let $m > n \geq n_0$. Then for all $a \in X$ and $t > 0$ we have

$$\begin{aligned} &M(y_m, y_n, a, t) \\ &\quad \geq \triangle(M(y_{n+1}, y_n, a, 3^{-1}t), \triangle(M(y_{n+1}, y_n, y_m, 3^{-1}t), \\ &\qquad M(y_{n+1}, y_m, a, 3^{-1}t))) \\ &\quad \geq \triangle(\triangle^2((1-\lambda), M(y_{n+1}, y_m, a, 3^{-1}t)). \end{aligned} \tag{3.3}$$

Since

$$\begin{aligned} &M(y_{n+1}, y_m, a, 3^{-1}t) \\ &\quad \geq \triangle(M(y_{n+2}, y_{n+1}, a, 3^{-2}t), \triangle(M(y_{n+2}, y_{n+1}, y_m, 3^{-2}t), \\ &\qquad M(y_{n+2}, y_m, a, 3^{-2}t))), \end{aligned}$$

from (3.3) we get

$$M(y_m, y_n, a, t) \geq \triangle(\triangle^{2^2}(1-\lambda), M(y_{n+2}, y_m, a, 3^{-2}t).$$

Inductively, we obtain

$$\begin{aligned} M(y_m, y_n, a, t) &\geq \triangle(\triangle^{2^{m-n}}(1-\lambda), M(y_m, y_m, a, 3^{n-m}t)) \\ &= \triangle^{2^{m-n}}(1-\lambda). \end{aligned} \tag{3.4}$$

From (3.2) and (3.4) we get for all $a \in X$ and $t > 0$ $M(y_m, y_n, a, t) > 1 - \epsilon$ for $m > n \geq n_0$. Thus $\{y_n\}$ is a Cauchy sequence.
It follows from completeness of X that there exists $z \in X$ such that $\lim_{n\to\infty} y_n = z$. Hence $\lim_{n\to\infty} y_{2n-1} = \lim_{n\to\infty} Tx_{2n-1} = \lim_{n\to\infty} Ax_{2n-2} = z$ and $\lim_{n\to\infty} y_{2n} = \lim_{n\to\infty} Sx_{2n} = \lim_{n\to\infty} Bx_{2n-1} = z$.
From Lemma 3.4, $ASx_{2n+1} \to Sz$ and $BTx_{2n+1} \to Tz$. (3.5)

Meanwhile, for all $a \in X$ with $a \neq Sz$ and $a \neq Tz$, and $t > 0$

$$\begin{aligned} &M(ASx_{2n+1}, BTx_{2n+1}, a, qt) \\ &\quad \geq \min\{M(SSx_{2n+1}, TTx_{2n+1}, a, t), M(ASx_{2n+1}, SSx_{2n+1}, a, t), \\ &\qquad M(BTx_{2n+1}, TTx_{2n+1}, a, t), M(ASx_{2n+1}, TTx_{2n+1}, a, t)\}. \end{aligned}$$

Taking limit as $n \to \infty$, and using (3.5), and Lemma 3.5, we have for all $a \in X$ with $a \neq Sz$ and $a \neq Tz$, and $t > 0$

$$\begin{aligned} &M(Sz, Tz, a, qt + 0) \\ &\quad \geq \min\{M(Sz, Tz, a, t), M(Sz, Sz, a, t), M(Tz, Tz, a, t), \\ &\qquad M(Sz, Tz, a, t)\} \\ &\quad \geq M(Sz, Tz, a, t). \end{aligned}$$

By Lemma 3.2, we have $Sz = Tz$ (3.6)

From (3.0) we get for all $a \in X$ with $a \neq Az$ and $a \neq Tz$, and $t > 0$

$$\begin{aligned} &M(Az, BTx_{2n+1}, a, qt) \\ &\quad \geq \min\{M(Sz, TTx_{2n+1}, a, t), M(Az, Sz, a, t), \\ &\qquad M(BTx_{2n+1}, TTx_{2n+1}, a, t), M(Az, TTx_{2n+1}, a, t)\}. \end{aligned}$$

Taking limit as $n \to \infty$, and using (3.5), (3.6) and Lemma 3.3,

$$\begin{aligned} &M(Az, Tz, a, qt + 0) \\ &\quad \geq \min\{M(Sz, Tz, a, t), M(Az, Sz, a, t), M(Tz, Tz, a, t), \\ &\qquad M(Az, Tz, a, t)\} \\ &\quad \geq M(Az, Tz, a, t). \end{aligned}$$

By Lemma 3.2, $Az = Tz.$ (3.7)
And for all $a \in X$ with $a \neq Az$ and $a \neq Bz$, and $t > 0$

$$\begin{aligned} &M(Az, Bz, a, qt) \\ &\quad \geq \min\{M(Sz, Tz, a, t), M(Az, Sz, a, t), M(Bz, Tz, a, t), \\ &\qquad M(Az, Tz, a, t)\} \\ &\quad \geq \min\{M(Tz, Tz, a, t), M(Tz, Tz, a, t), M(Bz, Az, a, t), \\ &\qquad M(Tz, Tz, a, t)\} \\ &\quad \geq M(Az, Bz, a, t). \end{aligned}$$

By Lemma 3.2, $Az = Bz.$ (3.8)
It follows that $Az = Bz = Sz = Tz.$
For all $a \in X$ with $a \neq Bz$ and $a \neq z$, and $t > 0$

$$\begin{aligned} &M(Ax_{2n}, Bz, a, qt) \\ &\quad \geq \min\{M(Sx_{2n}, Tz, a, t), M(Ax_{2n}, Sx_{2n}, a, t), \\ &\qquad M(Bz, Tz, a, t), M(Ax_{2n}, Tz, a, t)\}. \end{aligned}$$

Taking limit as $n \to \infty$, and using (3.5), and Lemma 3.3, we have for all $a \in X$ with $a \neq Bz$ and $a \neq z$, and $t > 0$

$$\begin{aligned} &M(z, Bz, a, qt + 0) \\ &\quad \geq \min\{M(z, Tz, a, t), M(z, z, a, t), M(Bz, Bz, a, t), M(z, Tz, a, t)\} \\ &\quad \geq M(z, Tz, a, t) \geq M(z, Bz, a, t), \end{aligned}$$

and so we have $M(z, Bz, a, qt) \geq M(z, Bz, a, t)$, and hence $Bz = z$.
Thus $z = Az = Bz = Sz = Tz$, and so z is a common fixed point of A, B, S and T.
For uniqueness, let w be another common fixed point of A, B, S and T.
Then, for all $a \in X$ with $a \neq z$ and $a \neq w$, and $t > 0$,

$$\begin{aligned} &M(z, w, a, qt) = M(Az, Bw, a, qt) \\ &\quad \geq \min\{M(Sz, Tw, a, t), M(Az, Sz, a, t), M(Bw, Tw, a, t), \\ &\qquad M(Az, Tw, a, t)\} \\ &\quad \geq \min\{M(z, w, a, t), M(z, z, a, t), M(w, w, a, t), M(z, w, a, t)\} \\ &\quad \geq M(z, w, a, t) \end{aligned}$$

which implies that $M(z, w, a, qt) \geq M(z, w, a, t)$ and hence $z = w$.
This complete the proof of Theorem. □

In Theorem 3.5, if we have $A = B$ then we obtain the following result.

Corollary 3.6. *Let $(X, M, \triangle)$ be a complete fuzzy 2-metric space with continuous t-norm $\triangle$ of H-type, and let S and T be continuous self mappings of X.*
Then S and T have a unique common fixed point in X if and only if there exists self mapping A of X satisfying

(1) $AX \subset TX \cap SX$,
(2) *the pair $[A, S]$ and $[A, T]$ are compatible,*
(3) *there exists $q \in (0, 1)$ such that for every $x, y, a \in X$ and $t > 0$,*
$$M(Ax, Ay, a, qt) \geq \min\{M(Sx, Ty, a, t), M(Ax, Sx, a, t), M(Ay, Ty, a, t), M(Ax, Ty, a, t)\}.$$

Indeed, A, S and T have a unique common fixed point in X.

We modify theorem 2 of [11] as the following.

Corollary 3.7. *(Sharma[11]) Let $(X, M, \triangle)$ be a complete fuzzy 2-metric space with continuous t-norm $\triangle$ of H-type, and let S and T be continuous self mappings of X.*
Then S and T have a unique common fixed point in X if and only if there exists self mapping A of X satisfying

(1) $AX \subset TX \cap SX$,
(2) *A commute with S and T,*
(3) *there exists $q \in (0, 1)$ such that for every $x, y, a \in X$ and $t > 0$,*
$$M(Ax, Ay, a, qt) \geq \min\{M(Sx, Ty, a, t), M(Ax, Sx, a, t), M(Ay, Ty, a, t)\}.$$

Indeed, A, S and T have a unique common fixed point in X.

In Corollary 3.7, if we have $S = T = id$ then we obtain the following result, where id is the identity map on X.

Corollary 3.8. *Let $(X, M, \triangle)$ be a complete fuzzy 2-metric space with continuous t-norm $\triangle$ of H-type, and let A be a self mapping of X satisfying*
there exists $q \in (0, 1)$ such that for every $x, y, a \in X$ and $t > 0$,

$$M(Ax, Ay, a, qt) \geq \min\{M(Sx, Ty, a, t), M(Ax, Sx, a, t), M(Ay, Ty, a, t)\}.$$

Then A has a unique common fixed point in X.

The following result is fuzzy 2-metric space version of Banach contraction principle.

COROLLARY 3.9. *Let $(X, M, \triangle)$ be a complete fuzzy 2-metric space with continuous t-norm $\triangle$ of H-type, and let A be a self mapping of X satisfying*
there exists $q \in (0,1)$ such that for every $x, y, a \in X$ and $t > 0$,
$M(Ax, Ay, a, qt) \geq M(x, y, a, t)$.
Then A has a unique common fixed point in X.

Rao et al. [9] point out that there are some errors in the paper of Cho [2]. They claimed that the conditions of fixed point theorems given in [2] are incorrect "in view of the example even when $S = T = id$, where *id* is the identity map". But the given examples in [9] are not satisfied the conditions of the theorems of Cho [2]. So, Rao's claim in [9] is inappropriate.

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