JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **23**, No. 4, December 2010

A SHARP LOWER BOUND OF THE FIRST NEUMANN EIGENVALUE OF A COMPACT HYPERSURFACE INSIDE A CONVEX SET

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Abstract. In this paper we provide a sharp lower bound of the first Neumann eigenvalue of a compact hypersurface Σ inside a convex set C in a Riemannian manifold under the assumption that $\partial\Sigma$ meets ∂C orthogonally.

1. Introduction

Let Σ be an n-dimensional compact Riemannian manifold with boundary $\partial\Sigma$. In terms of local coordinates $(x^1, \cdots, x^n)$, the Riemannian metric is given by $ds^2 = \sum_{i,j=1}^{n} g_{ij} dx^i dx^j$ and the Laplace operator is defined by

$$\Delta = \frac{1}{\sqrt{g}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x^i}(\sqrt{g} g^{ij} \frac{\partial}{\partial x^j}),$$

where $(g^{ij}) = (g_{ij})^{-1}$ and $g = \det(g_{ij})$. We shall deal with the following Neumann eigenvalue problem on Σ.

$$\begin{aligned} \Delta f + \lambda f &= 0 \qquad \text{in } \Sigma, \\ \frac{\partial f}{\partial \nu} &= 0 \qquad \text{on } \partial\Sigma, \end{aligned}$$

where ν is the unit outward normal vector to the boundary $\partial\Sigma$. The first nonzero eigenvalue λ_1 in the above Neumann eigenvalue problem is

Received May 12, 2010; Accepted August 12, 2010.
2010 Mathematics Subject Classification: Primary 58J50; Secondary 35P15.
Key words and phrases: Neumann eigenvalue, Laplacian operator, convex set.

characterized as follows.

$$\lambda_1 = \inf_{\phi \in H_1^2(\Sigma)} \frac{\int_\Sigma |\nabla \phi|^2}{\int_\Sigma \phi^2},$$

for all $\phi \in C^\infty(\Sigma)$ with $\int_\Sigma \phi = 0$.

For compact Riemannian manifolds with convex boundary, the estimates of the lower bound of λ_1 were obtained by Li-Yau [4] and Escobar [3]. In [1], R. Chen gave a lower bound of λ_1 for compact Riemannian manifolds with possibly nonconvex boundary.

In this paper, we treat the Neumann eigenvalue problem on Σ in a more geometric way. Let C be a convex body in an $(n+1)$-dimensional Riemannian manifold M. Let Σ be an immersed hypersurface in C which is smooth up to its boundary $\partial\Sigma$ and suppose that $\partial\Sigma$ meets ∂C orthogonally. For such hypersurface Σ, we obtain an estimate of the first Neumann eigenvalue as follows.

Theorem 1.1. *Let C be an $(n+1)$-dimensional convex subset of a Riemannian manifold M^{n+1} with boundary ∂C. Let Σ be a compact hypersurface in C whose boundary meets ∂C perpendicularly. Assume that* $\mathrm{Ric}_\Sigma \geq k(n-1)$ *for a positive constant k. Then the first Neumann eigenvalue λ_1 of the Laplacian of Σ satisfies*

$$\lambda_1 \geq nk.$$

Moreover, equality holds if and only if Σ is isometric to a hemisphere of radius $\frac{1}{\sqrt{k}}$ in the $(n+1)$-dimensional Euclidean space.

2. Proof of the main theorem

Let M be an n-dimensional Riemannian manifold with boundary ∂M. Let f be a function defined on Σ which is smooth up to ∂M. Let $\overline{\Delta} f$ and $\overline{\nabla} f$ denote the Laplacian and the gradient of f with respect to the Riemannian metric of M, whereas Δf and ∇f denote the Laplacian and the gradient of f with respect to the induced Riemannian metric on ∂M, respectively. For $p \in M$ and $X, Y \in T_pM$, the Hessian tensor is defined by $(\overline{D}^2 f)(X,Y) = X(Yf) - (\overline{\nabla}_X Y)f$, where $\overline{\nabla}_X Y$ is the covariant derivative of the Riemannian connection of M. Denote $z = f|_{\partial M}$ and $u = \frac{\partial f}{\partial \nu}\Big|_{\partial M}$, where $\frac{\partial f}{\partial \nu}$ is the outward normal derivative of f.

Let $\{e_1, \cdots, e_{n-1}, e_n\}$ be a local orthonormal frame such that $\{e_1, \cdots, e_{n-1}\}$ are tangent to ∂M and $e_n = \nu$ is the outward normal vector at $q \in \partial M$. The second fundamental form of ∂M in M is defined as

$\Pi(v, w) = \langle \overline{\nabla}_v e_n, w\rangle$, where v and w are vectors tangent to ∂M and the mean curvature H is given by $H = \sum_{i=1}^{n-1} \Pi(e_i, e_i)$. In order to prove our main theorem, we need the following well known formula which is called Reilly formula [5](See also [2]).

THEOREM 2.1 (Reilly formula).

$$\begin{aligned}\int_M (\overline{\Delta} f)^2 - |\overline{D}^2 f|^2 &= \int_M \mathrm{Ric}(\overline{\nabla} f, \overline{\nabla} f)\\ &+ \int_{\partial M} (\Delta z + Hu)u - \int_{\partial M} \langle \nabla z, \nabla u\rangle + \int_{\partial M} \Pi(\nabla z, \nabla z),\end{aligned} \tag{2.1}$$

where $\mathrm{Ric}(,)$ *is the Ricci tensor of* M.

We are now ready to prove our main theorem.

THEOREM 2.2. *Let* C *be an* $(n+1)$*-dimensional convex subset of a Riemannian manifold* M^{n+1} *with boundary* ∂C. *Let* Σ *be a compact hypersurface in* C *whose boundary meets* ∂C *perpendicularly. Assume that* $\mathrm{Ric}_\Sigma \geq k(n-1)$ *for a positive constant* k. *Then the first Neumann eigenvalue* λ_1 *of the Laplacian of* Σ *satisfies*

$$\lambda_1 \geq nk.$$

Moreover, equality holds if and only if Σ *is isometric to a hemisphere of radius* $\frac{1}{\sqrt{k}}$ *in the* $(n+1)$*-dimensional Euclidean space.*

Proof. Let f be the first eigenfunction on Σ, i.e.,

$$\overline{\Delta} f + \lambda_1 f = 0 \qquad \text{on } \Sigma,$$
$$u = \frac{\partial f}{\partial \nu} = 0 \qquad \text{on } \partial\Sigma,$$

where ν is the unit outward normal vector to the boundary $\partial\Sigma$. By the Cauchy-Schwarz inequality, one sees that $(\overline{\Delta} f)^2 \leq n|\overline{D}^2 f|^2$. Using this in the Reilly formula (2.1) and the fact that $u = \dfrac{\partial f}{\partial \nu} = 0$ on $\partial\Sigma$, we get

$$\begin{aligned}\int_\Sigma \frac{n-1}{n}(\overline{\Delta} f)^2 &\geq \int_\Sigma \mathrm{Ric}(\overline{\nabla} f, \overline{\nabla} f) + \int_{\partial\Sigma} \Pi(\nabla z, \nabla z)\\ &\geq k(n-1)\int_\Sigma |\overline{\nabla} f|^2 + \int_{\partial\Sigma} \Pi(\nabla z, \nabla z),\end{aligned}$$

where we used our assumption on the Ricci tensor in the last inequality.

Putting $\overline{\Delta} f = -\lambda_1 f$ into the above inequality, we get

$$\Big(\frac{n-1}{n}\Big)\lambda_1^2 \int_\Sigma f^2 \geq k(n-1)\int_\Sigma |\overline{\nabla} f|^2 + \int_{\partial\Sigma} \Pi(\nabla z, \nabla z). \tag{2.2}$$

Now we recall that the second fundamental form $\widetilde{\Pi}$ of ∂C in M is given by

$$\widetilde{\Pi}(V, W) = \langle \widetilde{\nabla}_V e_n, W \rangle,$$

where $\widetilde{\nabla}$ denotes the connection of M and V, W are vectors tangent to ∂C. Then the convexity of C implies that

$$\widetilde{\Pi}(V, V) = \langle \widetilde{\nabla}_V e_n, V \rangle = -\langle \widetilde{\nabla}_V V, e_n \rangle \geq 0 \tag{2.3}$$

for all $V \in T(\partial C)$.

We choose a unit vector e_{n+1} satisfying that $\{e_1, \cdots, e_n = \nu, e_{n+1}\}$ is a local orthonormal frame in M^{n+1}. It follows that e_{n+1} is perpendicular to $\partial\Sigma$, since $\partial\Sigma$ meets ∂C orthogonally. Given $v \in T(\partial\Sigma) \subset T(\partial C)$, we have

$$\widetilde{\nabla}_v v - \overline{\nabla}_v v \in N(\Sigma),$$

where $N(\Sigma)$ denotes the normal space of Σ. Hence we get $\langle \overline{\overline{\nabla}}_v v - \overline{\nabla}_v v, e_n \rangle = 0$, i.e.,

$$\langle \widetilde{\nabla}_v v, e_n \rangle = \langle \overline{\nabla}_v v, e_n \rangle. \tag{2.4}$$

It follows from (2.3) and (2.4) that

$$\Pi(v, v) = -\langle \overline{\nabla}_v v, e_n \rangle = \langle \widetilde{\nabla}_v v, e_n \rangle \geq 0.$$

Thus the inequality (2.2) becomes

$$\Big(\frac{n-1}{n}\Big)\lambda_1^2 \int_\Sigma f^2 \geq k(n-1) \int_\Sigma |\overline{\nabla} f|^2.$$

Since $\lambda_1 = \inf\limits_{\phi \in H_1^2(\Sigma)} \dfrac{\int_\Sigma |\overline{\nabla}\phi|^2}{\int_\Sigma \phi^2}$ for all $\phi \in C^\infty(\Sigma)$ satisfying that $\int_\Sigma \phi = 0$, we see that

$$\Big(\frac{n-1}{n}\Big)\lambda_1^2 \geq k(n-1)\frac{\int_\Sigma |\overline{\nabla} f|^2}{\int_\Sigma f^2} \geq k(n-1)\lambda_1.$$

Hence we obtain

$$\lambda_1 \geq nk.$$

If equality holds, then using Escobar's result [3, Theorem 4.2 and 4.3], we get Σ is isometric to a hemisphere of radius $\frac{1}{\sqrt{k}}$ in the $(n+1)$-dimensional Euclidean space. $\square$

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