JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **23**, No. 3, September 2010

ON HILBERT 2-CLASS FIELD TOWERS OF REAL QUADRATIC FUNCTION FIELDS

Hwanyup Jung*

Abstract. In this paper we prove that real quadratic function field F over $\mathbb{F}_q(T)$ has infinite 2-class field tower if the 4-rank of narrow ideal class group of F is equal to or greater than 4 when $q \equiv 3 \bmod 4$.

1. Introduction and statement of main result

Let $\mathrm{k} = \mathbb{F}_q(T)$ be the rational function field over the finite field $\mathbb{F}_q$ of q elements and $\mathbb{A} = \mathbb{F}_q[T]$. Let ∞ be the prime of k associated to $(1/T)$. For a finite separable extension F of k, let $\mathcal{O}_F$ be the integral closure of $\mathbb{A}$ in F and H_F be the Hilbert class field of F with respect to $\mathcal{O}_F$ ([5]). Let ℓ be a prime number. Let $F_1^{(\ell)}$ be the Hilbert ℓ-class field of $F_0^{(\ell)} = F$ (i.e., $F_1^{(\ell)}$ is the maximal ℓ-extension of F inside H_F) and inductively, $F_{n+1}^{(\ell)}$ be the Hilbert ℓ-class field of $F_n^{(\ell)}$ for $n \geq 1$. Then we obtain a sequence of fields $F_0^{(\ell)} = F \subset F_1^{(\ell)} \subset \cdots \subset F_n^{(\ell)} \subset \cdots$, which is called the *Hilbert ℓ-class field tower of* F. We say that the Hilbert ℓ-class field tower of F is infinite if $F_n^{(\ell)} \neq F_{n+1}^{(\ell)}$ for each $n \geq 0$. For any multiplicative abelian group A, write $r_\ell(A) = \dim_{\mathbb{F}_\ell}(A/A^\ell)$, which is called the the ℓ-rank of A. In [6], Schoof has proved that the Hilbert ℓ-class field tower of F is infinite if

$$r_\ell(\mathcal{C}(F)) \geq 2 + 2\sqrt{r_\ell(\mathcal{O}_F^*) + 1}, \tag{1.1}$$

Received July 07, 2010; Accepted August 12, 2010.
2010 Mathematics Subject Classification: Primary 11R29, 11R37, 11R58.
Key words and phrases: class field tower, real quadratic function field.
This research was supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education, Science and Technology(2010-0008139).

where $\mathcal{C}(F)$ and $\mathcal{O}_F^*$ are the ideal class group and the group of units of $\mathcal{O}_F$, respectively. This is a function field analog of the theorem of Golod-Shafarevich.

Assume that q is odd. Throughout the paper, by a *real quadratic function field*, we always mean a quadratic extension F of k in which ∞ splits. Any real quadratic function field F can be written uniquely as $F = \mathrm{k}(\sqrt{D})$, where $D \in \mathbb{A}$ is a nonconstant square-free monic polynomial of even degree. In this paper, we study the infiniteness of Hilbert 2-class field tower of such a real quadratic function field F. Since $\mathcal{O}_F^* \cong \mathbb{F}_q^* \times \mathbb{Z}$, $r_2(\mathcal{O}_F^*) = 2$, so the Hilbert 2-class field tower of F is infinite if $r_2(\mathcal{C}(F)) \geq 6$ by Schoof's theorem. Let $\mathcal{C}^+(F)$ be the narrow ideal class group of $\mathcal{O}_F$ (cf. §2.1). Write $r_4(\mathcal{C}^+(F)) = r_2(\mathcal{C}^+(F)^2)$, which is called the 4-rank of $\mathcal{C}^+(F)$. The main result of this paper is the following theorem.

THEOREM 1.1. *Assume that* $q \equiv 3 \bmod 4$. *Let* F *be a real quadratic function field over* k. *If* $r_4(\mathcal{C}^+(F)) \geq 4$, *then* F *has infinite Hilbert 2-class field tower.*

In classical case, Lemmermeyer [3] has proved a similar result for the real quadratic number field F. Our method is elementary since we only use the Rédei matrix M_F^+ associated to F and this method also works for real quadratic number field case.

2. Preliminaries

2.1. Narrow ideal class group $\mathcal{C}^+(F)$

Let $\mathrm{k}_\infty = \mathbb{F}_q((1/T))$ be the completion of k at ∞. Let $sgn : \mathrm{k}_\infty^* \to \mathbb{F}_q^*$ be the sign function satisfying $sgn(1/T) = 1$ and define $s(x) = sgn(x)^{\frac{q-1}{2}}$ for any $x \in \mathrm{k}_\infty^*$. Let F be a real quadratic function field over k. Let ∞_1 and ∞_2 be primes of F lying above ∞. Define a homomorphism

$$\mathbf{s} : F^* \to \{\pm 1\} \times \{\pm 1\},\ x \mapsto (s_1(x), s_2(x)),$$

where $s_i(x) = s(\eta_i(x))$ and η_i is the embedding of F into k_∞ associated to ∞_i for $i = 1, 2$. An element $x \in F^*$ is said to be *positive* if $\mathbf{s}(x) = (1, 1)$. Put $F^+ = \mathrm{Ker}(\mathbf{s})$, which is the subgroup of F^* consisting of all positive elements of F^*. Let $I(F)$ be the group of fractional ideals of $\mathcal{O}_F$ and $P^+(F)$ be the subgroup of $I(F)$ consisting of principal ideals generated by an element of F^+. The narrow ideal class group $\mathcal{C}^+(F)$ of $\mathcal{O}_F$ is defined as $\mathcal{C}^+(F) = I(F)/P^+(F)$.

2.2. 4-rank of $\mathcal{C}^+(F)$ and Rédei matrix M_F^+

Consider a real quadratic function field $F = \mathrm{k}(\sqrt{D})$ with $D = P_1 \cdots P_t$, where P_i is a monic irreducible polynomial in $\mathbb{A}$ for $1 \le i \le t$. By genus theory, $r_2(\mathcal{C}^+(F)) = t-1$. Let $s = s(D)$ denote the number of the P_i with odd degree. Since $\deg(D)$ is even, s is even. From now on we always assume that $2 \nmid \deg(P_i)$ for $1 \le i \le s$ and $2 | \deg(P_i)$ for $s+1 \le i \le t$. For $1 \le i \ne j \le t$, let $e_{ij} \in \mathbb{F}_2$ be defined by $(-1)^{e_{ij}} = (\frac{\bar{P}_i}{P_j})$, where $\bar{P}_i = (-1)^{\deg(P_i)} P_i$ and e_{ii} is defined to satisfy $\sum_{i=1}^t e_{ij} = 0$. Let $M'_F = (e_{ij})_{1\le i,j \le t}$. We associate a matrix M_F^+ to F defined as follows: If there is an ideal $\mathfrak{a}$ of F such that $\mathfrak{a}^{1-\sigma} = \alpha \mathcal{O}_F$ with $N_{F/\mathrm{k}}(\alpha) \in \mathbb{F}_q^{*2} \setminus \mathbb{F}_q^{*4}$, where σ is the generator of $\mathrm{Gal}(F/\mathrm{k})$, then M_F^+ is defined as the $t \times (t+1)$ matrix obtained from M'_F by adjoining $(e_{1A}\ e_{2A}\ \cdots\ e_{tA})^t$ to the last column, where $A \in \mathbb{A}$ is the monic polynomial with $N_{F/\mathrm{k}}(\mathfrak{a}) = (A)$ and $e_{iA} \in \mathbb{F}_2$ is defined to satisfy $(-1)^{e_{iA}} = (\frac{P_i}{A})$, and $M_F^+ = M'_F$ otherwise. We remark that if $q \equiv 3 \bmod 4$, we always have $M_F^+ = M'_F$. Then $r_4(\mathcal{C}^+(F))$ satisfies the following equality ([1])

$$r_4(\mathcal{C}^+(F)) = t - 1 - \mathrm{rank}(M_F^+). \tag{2.1}$$

2.3. Martinet's inequality

Let E and K be finite (geometric) separable extensions of k such that E/K is a cyclic extension of degree ℓ with $\Delta = \mathrm{Gal}(E/K)$, where ℓ is a prime number not dividing q. Then $H^0(\Delta, \mathcal{O}_E^*)$ and $H^1(\Delta, \mathcal{O}_E^*)$ are elementary abelian ℓ-groups with

$$\frac{|H^0(\Delta, \mathcal{O}_E^*)|}{|H^1(\Delta, \mathcal{O}_E^*)|} = \ell^{-1} \prod_{\mathfrak{p}_\infty \in S_\infty(K)} |\Delta_{\mathfrak{p}_\infty}|,$$

where $S_\infty(K)$ is the set of primes of K lying above ∞ and $\Delta_{\mathfrak{p}_\infty}$ denotes the decomposition group of $\mathfrak{p}_\infty$ in Δ. Note that $\Delta_{\mathfrak{p}_\infty} = \Delta$ if $\mathfrak{p}_\infty$ ramifies or inerts in E and $\Delta_{\mathfrak{p}_\infty} = \{1\}$ otherwise. Following the arguments in [4, §2], we get the following.

Proposition 2.1. *Let E and K be finite (geometric) separable extensions of* k *such that E/K is a cyclic extension of degree ℓ, where ℓ is a prime number not dividing q. Let $\gamma_{E/K}$ be the number of prime ideals of $\mathcal{O}_K$ that ramify in E and $\rho_{E/K}$ be the number of primes $\mathfrak{p}_\infty$ in $S_\infty(K)$ that ramify or inert in E. If $\gamma_{E/K}$ satisfies the inequality*

$$\gamma_{E/K} \ge |S_\infty(K)| - \rho_{E/K} + 3 + 2\sqrt{\ell |S_\infty(K)| + (1-\ell)\rho_{E/K} + 1}, \tag{2.2}$$

then the Hilbert ℓ-class field tower of E is infinite.

The inequality (2.2) is called the Martinet's inequality for E/K. Let F be a real quadratic function field over k. We remark that if there exists an extension E of F which has infinite Hilbert 2-class field tower and $F \subset E \subset F_1^{(2)}$, then F also has infinite Hilbert 2-class field tower.

Corollary 2.2. *Let $F = \mathrm{k}(\sqrt{D})$ be a real quadratic function field over* k*. If D has a nonconstant monic divisor D_1 of even degree satisfying $(\frac{D_1}{Q_j}) = 1$ for monic irreducible divisors Q_j $(1 \le j \le 5)$ of D, then F has infinite Hilbert 2-class field tower.*

Proof. Put $K = \mathrm{k}(\sqrt{D_1})$, which is a real quadratic extension of k in which Q_1, Q_2, Q_3, Q_4 and Q_5 split. Let $E = KF$. Applying Proposition 2.1 on E/K with $\gamma_{E/K} = 10$ and $(|S_\infty(K)|, \rho_{E/K}) = (2, 0)$, we see that E has infinite Hilbert 2-class field tower, so F also has infinite Hilbert 2-class field tower. □

Corollary 2.3. *Let $F = \mathrm{k}(\sqrt{D})$ be a real quadratic function field over* k*. If D has a two distinct nonconstant monic divisors D_1 and D_2 of even degrees satisfying $(\frac{D_1}{Q_j}) = (\frac{D_2}{Q_j}) = 1$ for monic irreducible divisors Q_j $(1 \le j \le 4)$ of D, then F has infinite Hilbert 2-class field tower.*

Proof. Put $K = \mathrm{k}(\sqrt{D_1}, \sqrt{D_2})$, which is a real biquadratic extension of k in which Q_1, Q_2, Q_3 and Q_4 split completely. Let $E = KF$. Applying Proposition 2.1 on E/K with $\gamma_{E/K} \ge 16$ and $(|S_\infty(K)|, \rho_{E/K}) = (4, 0)$, we see that E has infinite Hilbert 2-class field tower, so F also has infinite Hilbert 2-class field tower. □

Corollary 2.4. *Let $F = \mathrm{k}(\sqrt{D})$ be a real quadratic function field over* k*. If D has a two distinct nonconstant monic divisors D_1 and D_2 of even degrees satisfying $(\frac{D_1}{Q_j}) = (\frac{D_2}{Q_j}) = 1$ for monic irreducible divisors Q_j $(1 \le j \le 3)$ of D and there is a monic irreducible divisor Q of D which is different from Q_1, Q_2, Q_3 and $Q \nmid D_1D_2$, then F has infinite Hilbert 2-class field tower.*

Proof. Put $K = \mathrm{k}(\sqrt{D_1}, \sqrt{D_2})$, which is a real biquadratic extension of k in which Q_1, Q_2 and Q_3 split completely. Let $E = KF$. Since Q splits in at least one quadratic subfield of K, we have $\gamma_{E/K} \ge 14$. Applying Proposition 2.1 on E/K with $\gamma_{E/K} \ge 14$ and $(|S_\infty(K)|, \rho_{E/K}) = (4, 0)$, we see that E has infinite Hilbert 2-class field tower, so F also has infinite Hilbert 2-class field tower. □

3. Proof of Theorem 1.1

Consider a real quadratic function field $F = \mathrm{k}(\sqrt{D})$ with $D = P_1 \cdots P_t$, where P_i is a monic irreducible polynomial in $\mathbb{A}$ for $1 \le i \le t$. Recall that $s = s(D)$ is the number of the P_i with odd degree and we assume that $2 \nmid \deg(P_i)$ for $1 \le i \le s$ and $2 | \deg(P_i)$ for $s+1 \le i \le t$. Assume that $q \equiv 3 \bmod 4$. In this section, we are going to prove the infiniteness of Hilbert 2-class field tower of F under the condition $r_4(\mathcal{C}^+(F)) \ge 4$. Note that $r_2(\mathcal{C}(F)) = t-1$ if $s = 0$ and $t-2$ if $s \ge 2$. Hence, if $r_2(\mathcal{C}^+(F)) \ge 6$ when $s = 0$ or $r_2(\mathcal{C}^+(F)) \ge 7$ when $s \ge 2$, then $r_2(\mathcal{C}(F)) \ge 6$, so F has infinite Hilbert 2-class field tower. If $r_2(\mathcal{C}^+(F)) = r_4(\mathcal{C}^+(F))$, then $\operatorname{rank} M_F^+ = 0$, i.e., $M_F^+ = O$, so $e_{12} = e_{21}$. Thus the case $r_2(\mathcal{C}^+(F)) = r_4(\mathcal{C}^+(F))$ with $s \ge 2$ can't occur by the quadratic reciprocity law. Thus we only need to consider the cases

$$(r_2(\mathcal{C}^+(F)), r_4(\mathcal{C}^+(F))) = \begin{cases} (4,4), (5,4), (5,5) & \text{if } s = 0, \\ (5,4), (6,4), (6,5) & \text{if } s \ge 2. \end{cases}$$

Let $\mathbf{r}_i(F)$ denote the i-th row of M_F^+ and $\mathbf{0}$ denote the zero one in $\mathbb{F}_2^t$.

• Case $(r_2(\mathcal{C}^+(F)), r_4(\mathcal{C}^+(F))) = (4,4)$ with $D = P_1P_2P_3P_4P_5$ and $s = 0$. Since $M_F^+ = O$, $(\frac{P_1}{P_i}) = (\frac{P_2}{P_i}) = 1$ for $3 \le i \le 5$, so P_3, P_4 and P_5 split completely in $K = \mathrm{k}(\sqrt{P_1}, \sqrt{P_2})$. Let $E = KF$. Since $\mathbb{F}_q^* = \mathbb{F}_q^* \cap N_{F/\mathrm{k}}(F^*)$, $\mathbb{F}_q^*$ is contained in $\mathcal{O}_K^* \cap N_{E/K}(E^*)$ and so $(\mathcal{O}_K^* : \mathcal{O}_K^* \cap N_{E/K}(E^*)) \le 2^3$. Since $(\frac{P_1}{P_2}) = 1$, the ideal class number $h(\mathcal{O}_K)$ of $\mathcal{O}_K$ is even. Since $\gamma_{E/K} = 12$, by the ambiguous class number formula ([2, Lemma 2.2]), $r_2(\mathcal{C}(E)) \ge 9$. Since $r_2(\mathcal{O}_E^*) = 8$, by Schoof's theorem, E has infinite Hilbert 2-class field tower. Since $F \subset E \subset F_1^{(2)}$, F also has infinite Hilbert 2-class field tower.

• Case $(r_2(\mathcal{C}^+(F)), r_4(\mathcal{C}^+(F))) = (5,5)$ with $D = P_1P_2P_3P_4P_5P_6$ and $s = 0$. Since $M_F^+ = O$, $(\frac{P_6}{P_i}) = 1$ for $1 \le i \le 5$, so F has infinite Hilbert 2-class field tower by Corollary 2.2.

• Case $(r_2(\mathcal{C}^+(F)), r_4(\mathcal{C}^+(F))) = (5,4)$ with $D = P_1P_2P_3P_4P_5P_6$ and $s \in \{0,2,4,6\}$. In this case $\operatorname{rank} M_F^+ = 1$, so at least one row of M_F^+ is nonzero and the other ones are multiple of this row. Assume first $s = 0$. Since at least two rows of M_F^+ are equal, we may assume $\mathbf{r}_5(F) = \mathbf{r}_6(F)$. Then $e_{5j} = e_{6j}$ for $1 \le j \le 4$, so P_1, P_2, P_3 and P_4 split in $K = \mathrm{k}(\sqrt{P_5P_6})$. Let $E = KF$. Since $\mathbb{F}_q^* = \mathbb{F}_q^* \cap N_{F/\mathrm{k}}(F^*)$, $\mathbb{F}_q^*$ is contained in $\mathcal{O}_K^* \cap N_{E/K}(E^*)$ and so $(\mathcal{O}_K^* : \mathcal{O}_K^* \cap N_{E/K}(E^*)) \le 2$. Since $\gamma_{E/K} = 8$ and $r_2(\mathcal{C}(K)) = 1$, by the ambiguous class number formula, $r_2(\mathcal{C}(E)) \ge 7$. Since $r_2(\mathcal{O}_E^*) = 4$, by (1.1), E has infinite Hilbert 2-class field tower. Since $F \subset E \subset F_1^{(2)}$, F also has infinite Hilbert 2-class field tower.

Assume that $s = 2$ or 4. If $\mathbf{r}_i(F) = \mathbf{0}$ for some $s+1 \leq i \leq 6$, say $\mathbf{r}_6(F) = \mathbf{0}$, then $(\frac{P_6}{P_j}) = 1$ for $1 \leq j \leq 5$, so F has infinite Hilbert 2-class field tower by Corollary 2.2. Now we may assume that $\mathbf{r}_i(F) \neq \mathbf{0}$ for $s+1 \leq i \leq 6$, so they are all equal. If $\mathbf{r}_i(F) = \mathbf{r}_j(F) = \mathbf{0}$ for some $1 \leq i \neq j \leq s$, say $\mathbf{r}_1(F) = \mathbf{r}_2(F) = \mathbf{0}$, then $e_{12} = e_{21} = 0$ which is a contradiction. Thus at most one row of M_F^+ is zero. Then all rows of M_F^+ are nonzero and they are all equal. If $e_{12} = 1$, then all rows of M_F^+ are (1 0 1 1 1 1), but then $e_{26} = 1 \neq 0 = e_{62}$ which is a contradiction. If $e_{12} = 0$, then all rows of M_F^+ are (0 1 0 0 0 0), but then $e_{26} = 0 \neq 1 = e_{62}$ which is a contradiction.

Consider the case $s = 6$. If $\mathbf{r}_i(F) = \mathbf{r}_j(F) = \mathbf{0}$ for some $1 \leq i \neq j \leq 6$, say $\mathbf{r}_1(F) = \mathbf{r}_2(F) = \mathbf{0}$, then $e_{12} = e_{21} = 0$ which is a contradiction. Thus at most one row of M_F^+ is zero. Then all rows of M_F^+ are nonzero and they are all equal, so we can get a contradiction as above. Thus this case can't occur.

• Case $(r_2(\mathcal{C}^+(F)), r_4(\mathcal{C}^+(F))) = (6, 4)$ with $D = P_1 \cdots P_7$ and $s \in \{2, 4, 6\}$. In this case, rank $M_F^+ = 2$, so two rows of M_F^+ are independent over $\mathbb{F}_2$ and the others are $\mathbb{F}_2$-linear combinations of these two rows. If $\mathbf{r}_7(F) = \mathbf{0}$, then $(\frac{P_7}{P_j}) = 1$ for $1 \leq j \leq 6$, so F has infinite Hilbert 2-class field tower by Corollary 2.2. Thus we may assume $\mathbf{r}_7(F) \neq \mathbf{0}$. Consider first the case $s = 2$. At least two of $\mathbf{r}_3(F), \mathbf{r}_4(F), \mathbf{r}_5(F), \mathbf{r}_6(F)$ and $\mathbf{r}_7(F)$ are equal, say $\mathbf{r}_6(F) = \mathbf{r}_7(F)$, then $(\frac{P_6P_7}{P_j}) = 1$ for $1 \leq j \leq 5$, so F has infinite Hilbert 2-class field tower by Corollary 2.2.

Assume $s = 4$. If two of $\mathbf{r}_1(F), \mathbf{r}_2(F), \mathbf{r}_3(F)$ and $\mathbf{r}_4(F)$ are equal, say $\mathbf{r}_1(F) = \mathbf{r}_2(F)$, then $(\frac{P_1P_2}{P_j}) = 1$ for $3 \leq j \leq 7$, so F has infinite Hilbert 2-class field tower by Corollary 2.2. Hence, we may assume that $\mathbf{r}_1(F), \mathbf{r}_2(F), \mathbf{r}_3(F)$ and $\mathbf{r}_4(F)$ are all distinct (†). If $\mathbf{r}_i(F) = \mathbf{0}$ for $i = 5$ or 6, say $\mathbf{r}_6(F) = \mathbf{0}$, then $(\frac{P_6}{P_j}) = 1$ for $1 \leq j \leq 5$, so F has infinite Hilbert 2-class field tower by Corollary 2.2. If two of $\mathbf{r}_5(F), \mathbf{r}_6(F)$ and $\mathbf{r}_7(F)$ are equal, say $\mathbf{r}_6(F) = \mathbf{r}_7(F)$, then $(\frac{P_6P_7}{P_j}) = 1$ for $1 \leq j \leq 5$, so F has infinite Hilbert 2-class field tower by Corollary 2.2. Thus, we may assume that $\mathbf{r}_5(F), \mathbf{r}_6(F)$ and $\mathbf{r}_7(F)$ are all distinct and nonzero (‡). From (†) and (‡), without loss of generality, we may assume that $\mathbf{r}_5(F), \mathbf{r}_6(F)$ are linearly independent over $\mathbb{F}_2$ and $\mathbf{r}_1(F) = \mathbf{0}$, $\mathbf{r}_2(F) = \mathbf{r}_5(F)$, $\mathbf{r}_3(F) = \mathbf{r}_6(F)$, $\mathbf{r}_4(F) = \mathbf{r}_7(F) = \mathbf{r}_5(F) + \mathbf{r}_6(F)$. But, since $\mathbf{r}_1(F) = \mathbf{0}$, we have $e_{21} = e_{31} = e_{41} = 1$ and $e_{51} = e_{61} = e_{71} = 0$, which is a contradiction.

Assume $s = 6$. Since $\mathbf{r}_7(F) \neq \mathbf{0}$, at least two rows of M_F^+ except the 7-th row are equal, say $\mathbf{r}_1(F) = \mathbf{r}_2(F)$. Then $(\frac{P_1P_2}{P_j}) = 1$ for $3 \leq j \leq 7$, so F has infinite Hilbert 2-class field tower by Corollary 2.2.

• Case $(r_2(\mathcal{C}^+(F)), r_4(\mathcal{C}^+(F))) = (6, 5)$ with $D = P_1 \cdots P_7$ and $s \in \{2, 4, 6\}$. In this case, rank $M_F^+ = 1$, so at least one row of M_F^+ is nonzero and the other ones are multiple of this row. If $\mathbf{r}_7(F) = \mathbf{0}$, then $(\frac{P_7}{P_j}) = 1$ for $1 \leq j \leq 6$, so F has infinite Hilbert 2-class field tower by Corollary 2.2. Thus we may assume $\mathbf{r}_7(F) \neq \mathbf{0}$ and the other rows of M_F^+ are multiple of $\mathbf{r}_7(F)$.

Assume that $s = 2$ or 4. If $\mathbf{r}_i(F) = \mathbf{0}$ for $i = 5$ or 6, say $\mathbf{r}_6(F) = \mathbf{0}$, then $(\frac{P_6}{P_j}) = 1$ for $1 \leq j \leq 5$, so F has infinite Hilbert 2-class field tower by Corollary 2.2. We may assume that $\mathbf{r}_i(F) \neq \mathbf{0}$ for $i = 5, 6$, so $\mathbf{r}_5(F) = \mathbf{r}_6(F) = \mathbf{r}_7(F)$. Then $(\frac{P_5P_6}{P_j}) = (\frac{P_5P_7}{P_j}) = 1$ for $1 \leq j \leq 4$, so F has infinite Hilbert 2-class field tower by Corollary 2.4.

Consider the case $s = 6$. At least three rows of M_F^+ except the 7-th row are equal, say $\mathbf{r}_4(F) = \mathbf{r}_5(F) = \mathbf{r}_6(F)$. Then $(\frac{P_4P_5}{P_j}) = (\frac{P_4P_6}{P_j}) = 1$ for $1 \leq j \leq 3$, so F has infinite Hilbert 2-class field tower by Corollary 2.4. We complete the proof of Theorem 1.1.

References

[1] S. Bae and H. Jung, *ℓ-rank of class groups of function fields.* submitted for publication.

[2] H. Jung, *Imaginary bicyclic function fields with the real cyclic subfield of class number one.* Bull. Korean Math. Soc. **45** (2008), no. 2, 375–384.

[3] F. Lemmermeyer, *The 4-class group of real quadratic number fields.* Preprint (1998).

[4] J. Martinet, *Tours de corps de classes et estimations de discriminants.* Invent. Math. **44** (1978), no. 1, 65–73.

[5] M. Rosen, *The Hilbert class field in function fields.* Exposition. Math. **5** (1987), no. 4, 365–378.

[6] R. Schoof, *Algebraic curves over $\mathbb{F}_2$ with many rational points.* J. Number Theory **41** (1992), no. 1, 6–14.

*
Department of Mathematics Education
Chungbuk National University
Cheongju 361-763, Republic of Korea
E-mail: hyjung@chungbuk.ac.kr