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SOME GEOMETRIC CONSEQUENCES OBTAINED FROM PARTIAL ELIMINATION IDEALS

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Abstract. In [9], M. Green introduced the partial elimination ideals defining the multiple loci of the projection image of a closed subscheme in $\mathbb{P}^n$. In this paper, we give some geometric consequences obtained from partial elimination ideals.

1. Introduction

Let V be a vector space of dimension $n+1$ over an algebraically closed field k of characteristic zero with basis $x_0, \dots, x_n$. If X is a nondegenerate reduced closed subscheme in $\mathbb{P}^n_k = \mathbb{P}(V)$ we write I_X for the saturated defining ideal of X in the coordinate ring $R = k[x_0, \dots, x_n]$ of $\mathbb{P}(V)$. If W is a subspace of V with a basis $x_t, \dots, x_n$ we write S_t for the symmetric algebra $\mathrm{Sym}(W) = k[x_t, x_{t+1} \dots, x_n]$. Let Λ be a linear subvariety in $\mathbb{P}^n_k = \mathbb{P}(V)$ with homogeneous coordinates $x_0, \dots, x_{t-1}$.

If we consider an outer projection of X from the center Λ

$$\pi_\Lambda : X \to \mathbb{P}^{n-t}_k = \mathbb{P}(W),$$

then the simplest question one could ask about the projection $\pi_\Lambda : X \to \mathbb{P}^{n-t}_k$ is the following: what can be said about the set of fibers?, or what sort of set is the image? These questions are the beginning of elimination theory (see [1], [2], [3], [5], [7], [9], [10]).

Partial elimination ideals which have been introduced by M. Green ([9]) can be used to study this kind of questions. Through the use of partial elimination ideals, these can be changed to questions about homogeneous ideals in polynomials rings (see [3], [4]).

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This paper is devoted to investigate some geometric consequences obtained from partial elimination ideals. We will focus on the following free presentation:

$$\bigoplus S_1(-j)^{\oplus\beta_{1,j}} \xrightarrow{\varphi_1} \bigoplus S_1(-j)^{\oplus\beta_{0,j}} \xrightarrow{\varphi_0} R/I_X \to 0.$$

We give a geometric meaning of the kernel of the map φ_0 (Theorem 3.4) by showing that the kernel of φ_0 is deeply related to partial elimination ideals (Proposition 3.3). These results show a relationship between partial elimination ideals and projection images of X. As an application, we recover that multiple locus of projections are defined by partial elimination ideals set-theoretically, which is given by M. Green in [9].

2. Preliminaries

In this section we recall some notations and definitions which will be used throughout the remaining part of the paper.

Let $R = k[x_0, \ldots, x_n]$ where k is an algebraically closed field of characteristic zero. For an element $\alpha = (\alpha_0, \ldots, \alpha_n) \in \mathbb{Z}^{n+1}_{\geq 0}$, we let x^α denote $x_0^{\alpha_0} \cdots x_n^{\alpha_r}$. Note that an ordering $>$ on $\mathbb{Z}^{n+1}_{\geq 0}$ gives us an ordering on monomials in R.

The graded lexicographic order (**grlex** order for short) is a typical example of orderings on n-tuples.

Definition 2.1. ([2], [3], [4]) Let α and β be elements in $\mathbb{Z}^n_{\geq 0}$. We say $\alpha >_{\text{grlex}} \beta$ if $\deg(x^\alpha) > \deg(x^\beta)$, or

(a) $\deg(x^\alpha) = \deg(x^\beta)$
(b) the leftmost nonzero entry of $\alpha - \beta$ is positive.

There is a notion of regularity for sheaves on projective spaces due to David Mumford that generalizes the idea of Castelnuovo. A closely related notion for graded modules arises naturally in the study of finite free resolutions and we present it here.

Definition 2.2. ([6], [7], [8]) For an $(n+1)$-dimensional k-vector space V with basis $x_0, \ldots, x_n$, we form the symmetric algebra $R = \mathrm{Sym}(V) = k[x_0, \ldots, x_n]$.
(a) For a finitely generated graded R-module $M = \bigoplus_{\ell \geq 0} M_\ell$, consider a minimal free resolution

$$\cdots \to \bigoplus_j R(-i-j)^{\beta_{i,j}(M)} \to \cdots \to \bigoplus_j R(-j)^{\beta_{0,j}(M)} \to M \to 0$$

of M as a graded R-modules. Thus $\beta^R_{i,j}(M) := \dim_k \operatorname{Tor}^R_i(M,k)_{i+j}$. We say that M is m-regular if $\beta_{i,j}(M) = 0$ for all $i \geq 0$ and $j \geq m$. The Castelnuovo-Mumford regularity of M is defined by

$$\operatorname{reg}(M) := \min\{\, m \mid M \text{ is } m\text{-regular}\}.$$

(b) For a coherent sheaf $\mathcal{M}$ on $\mathbb{P}(V)$, let $M = \bigoplus_{\ell\in\mathbb{Z}} H^0(\mathcal{M}(\ell))$ be its associated graded R-module. Then we write

$$\operatorname{reg}(\mathcal{M}) := \min\{\, m \mid H^i(\mathcal{M}(m-i)) = 0 \text{ for all } i \geq 1\}.$$

In this case, it is well known that $\operatorname{reg}(M) = \operatorname{reg}(\mathcal{M})$ (see [6]).

For a proof of main theorem, we need the following lemma.

Lemma 2.3. *If $0 \to A \to B \to C \to 0$ is a short exact sequence of graded finitely generated R-modules, then*

(a) $\operatorname{reg}(A) \leq \max\{\operatorname{reg}(B), \operatorname{reg}(C)+1\}$,
(b) $\operatorname{reg}(B) \leq \max\{\operatorname{reg}(A), \operatorname{reg}(C)\}$,
(c) $\operatorname{reg}(C) \leq \max\{\operatorname{reg}(A)-1, \operatorname{reg}(B)\}$.

Proof. See Corollary 20.19 in [7] for a proof. □

3. Partial elimination ideals

In this section we define the partial elimination ideals and describe their basic algebraic and geometric properties. Let $\pi_q : X \to Y \subset \mathbb{P}^{n-1}$ be an outer projection from the center $q = [1 : 0 : \cdots : 0]$. For the degree lexicographic order, if $f \in I_X$ has leading term $\operatorname{in}(f) = x_0^{d_0} \cdots x_n^{d_n}$, we set $d_0(f) = d_0$, the leading power of x_0 in f. Then it is well known that

$$I_Y = \bigoplus_{m\geq 0} \big\{ f \in (I_X)_m \mid d_0(f) = 0 \big\} = I_X \cap S.$$

More generally, one can define partial elimination ideals of I_X, which was given by M. Green in [9].

Definition 3.1 ([9]). Let $I_X \subset R$ be a homogeneous ideal of X and let

$$\tilde{K}_i(I_X) = \bigoplus_{m\geq 0} \big\{ f \in (I_X)_m \mid d_0(f) \leq i \big\}.$$

If $f \in \tilde{K}_i(I_X)$, we may write uniquely $f = x_0^i \bar{f} + g$ where $d_0(g) < i$. Now we define $K_i(I_X)$ by the image of $\tilde{K}_i(I_X)$ in S under the map $f \mapsto \bar{f}$ and we call $K_i(I_X)$ the i-th partial elimination ideal of I_X.

REMARK 3.2. (a) If we let $S = k[x_1, \ldots, x_n]$ then $\tilde{K}_k(I)$ and $K_k(I)$ are graded S-modules. Note that $\tilde{K}_k(I)$ is not a graded R-module in general.
(b) Let $P = (x_1, \ldots, x_n)$ be the defining ideal of a point $p = [1 : 0 : \cdots : 0]$. For a reduced closed subscheme X in $\mathbb{P}^n$, if we write I_X for the defining ideal of X then note that $f \in \tilde{K}_k(I_X)_d$ if and only if $f \in P^{d-k}$ if and only if $\mathrm{mult}_p(f) \geq d-k$. This result follows directly from the fact that

(i) $\mathrm{mult}_p(f)$ is the length of $R/(f) \otimes R_P$
(ii) the length of $(R/P^{d-k}) \otimes R_P$ is equal to $d-k$.

Proposition 3.3 and Theorem 3.4 are main results in this paper, which give a relationship between the partial elimination ideals and the geometry of the projection map from $\mathbb{P}^n$ to $\mathbb{P}^{n-1}$.

PROPOSITION 3.3. *Let X be a reduced closed subscheme in $\mathbb{P}^n$. Suppose that $\pi_q : X \to Y \subset \mathbb{P}^{n-1}$ be a projection from the center $q = [1 : 0 \cdots : 0]$. Then, as a S_1-module, there is a free presentation of R/I_X*

$$\bigoplus S_1(-j)^{\oplus \beta_{1,j}} \xrightarrow{\varphi_1} \bigoplus S_1(-j)^{\oplus \beta_{0,j}} \xrightarrow{\varphi_0} R/I_X \to 0,$$

such that the kernel of φ_0 is $\tilde{K}_d(I_X)$ for some $d > 0$.

Proof. Note that we can choose a homogeneous polynomial of the following form in the ideal I_X:

$$f = x_0^{d+1} + x_0^d g_d + \cdots + x_0 g_1 + g_0 \text{ for some } d \geq 0,$$

where g_i is a homogeneous form of degree $d-i+1$ in $S_1 = k[x_1, \ldots, x_n]$. This follows from the fact that $q \notin X$. From the definition of partial elimination ideals, we have the d-th partial elimination ideal $K_{d+1}(I_X)$ is $S_1 = k[x_1, \ldots, x_n]$. Consider a graded S_1-module homomorphism $\phi_0 : \oplus_{i=0}^d S(-i) \to R/I_X$ defined by $\phi_0(e_i) = x_0^i$ for each free basis e_i of $S(-i)$.

Now we claim that φ_0 is surjective and the kernel of φ_0 is $\tilde{K}_d(I_X)$. First, note that

$$x^{d+1} \equiv x_0^d g_d + \cdots + x_0 g_1 + g_0 \mod I_X.$$

Hence, this equation can be used to express every monomial x^m for $m \geq n$ modulo I_X in terms of monomials x^α, where $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_n)$ and $\alpha_0 \leq d$. This implies that the S_1-module homomorphism ϕ_0 is surjective.

Now let us prove $\ker\varphi_0 = \tilde{K}_d(I_X)$. It suffices to show that $\ker\varphi_0 \subset \tilde{K}_d(I_X)$ since $\tilde{K}_d(I_X) \subset I_X$ and thus $\varphi_0(\tilde{K}_d(I_X))$ is vanishing. Suppose that

$$G = (g_d, \dots g_1, g_0) \in \bigoplus_{i=0}^{d} S(-i)$$

is an element in the kernel of φ_0. Then $\varphi_0(G) = x_0^d g_d + \cdots + x_0 g_1 + g_0$ has to be contained in I_X and thus

$$\varphi_0(G) \in \tilde{K}_d(I_X).$$

Consequently, we construct a free presentation of R/I_X as a S_1-module

$$\bigoplus S_1(-j)^{\beta_{1,j}} \xrightarrow{\varphi_1} \bigoplus_{i=0}^{d} S_1(-j) \xrightarrow{\varphi_0} R/I_X \to 0,$$

and the kernel of φ_0 is $\tilde{K}_d(I_X)$, as we wished. □

THEOREM 3.4. *Let X be a reduced closed subscheme in $\mathbb{P}^n$ and we write I_X for the defining ideal of X. If L is a line through the point $p = [1 : 0 \cdots : 0]$ then we have*

$$L \subset Z\big(\tilde{K}_k(I_X)\big) \text{ if and only if } \operatorname{length}(L \cap X) > k.$$

Proof. ($\Leftarrow$): Suppose that $f \in \tilde{K}_k(I_X)_d$ and $p = [1 : 0 \cdots : 0]$. Then we have $f \in P^{d-k} = (x_1, \dots, x_n)^{d-k}$ and $\operatorname{mult}_p(f) \geq d-k$ by Remark 3.2. For a line L through the point p, if the length of intersections between X and L is at least $k+1$ then

$$\operatorname{length}(Z(f) \cap L) \geq \operatorname{mult}_p(f) + \operatorname{length}(X \cap L) \geq (d-k) + (k+1) = d+1.$$

Since f is a homogeneous polynomial of degree d, this implies that f is vanishing on L. Hence $L \subset Z\big(\tilde{K}_k(I_X)\big)$.

($\Rightarrow$) Conversely, suppose that there is a line $L \subset Z\big(\tilde{K}_k(I_X)\big)$ passing through the point p with $\operatorname{length}(X \cap L) \leq k$. Then it suffices to show that L is not contained in $Z\big(\tilde{K}_k(I_X)\big)$. This can be done if we prove that there is a polynomial $f \in \tilde{K}_k(I_X)_d$ such that f is not vanishing on the line L,

Now consider the following short exact sequence:

$$(3.1) \qquad 0 \to I_X \cap P^{d-k} \cap I_L \to I_X \cap P^{d-k} \to \frac{I_X \cap P^{d-k}}{I_X \cap P^{d-k} \cap I_L} \to 0,$$

Let $Y = X \cup p^{d-k}$ be the disjoint union of a fat point p^{d-k} and X. Since we have

$$\frac{I_X \cap P^{d-k}}{I_X \cap P^{d-k} \cap I_L} = \frac{I_Y}{I_Y \cap I_L} \cong \frac{(I_Y, I_L)}{I_L},$$

and we can think of the ideal $\frac{(I_Y,I_L)}{I_L}$ as the defining ideal of collinear zero dimensional subscheme on the line L, we conclude that

$$\begin{aligned} \operatorname{reg}\left(\frac{I_X \cap P^{d-k}}{I_X \cap P^{d-k} \cap I_L}\right) &= \operatorname{reg}\left(\frac{I_Y + I_L}{I_L}\right) \\ &\leq \deg(Y) \\ &\leq \operatorname{length}(X \cap L) + (d-k) \\ &\leq d. \end{aligned}$$

Since Y is the disjoint union of a fat point p^{d-k} and X, we have

$$\operatorname{reg}(Y) = \max\{ \operatorname{reg}(X), \operatorname{reg}(p^{d-k}) \} \leq d$$

for all sufficiently large integer d. Consequently, by Lemma 2.3, we have

$$\begin{aligned} \operatorname{reg}(Y \cup L) &\leq \max\{\operatorname{reg}(Y), \operatorname{reg}\left(\frac{I_X \cap P^{d-k}}{I_X \cap P^{d-k} \cap I_L}\right) + 1\} \\ &\leq d+1, \end{aligned}$$

and thus $H^1(\mathbb{P}^n, \mathcal{I}_{L \cup X \cup P^{d-k}}(d)) = 0$ for all $d \gg 0$.

By sheafifying (3.1), we have the following short exact sequence of sheaves

$$0 \to \mathcal{I}_{L\cup X\cup P^{d-k}} \to \mathcal{I}_{X\cup P^{d-k}} \to \mathcal{I}_{X\cup P^{d-k}/L\cup X\cup P^{d-k}} \to 0,$$

and we conclude that, for all $d \gg 0$,

$$H^0(\mathbb{P}^n, \mathcal{I}_{X\cup P^{d-k}}(d)) \to H^0(L \cup X \cup P^{d-k}, \mathcal{I}_{X\cup P^{d-k}}(d))$$

is surjective from the vanishing of $H^1(\mathbb{P}^n, \mathcal{I}_{L\cup X\cup P^{d-k}}(d)) = 0$. Now choose a nonzero form $\bar{f}$ in $H^0(L \cup X \cup P^{d-k}, \mathcal{I}_{X\cup P^{d-k}}(d))$. If we write $f \in H^0(\mathbb{P}^n, \mathcal{I}_{X\cup P^{d-k}}(d))$ for the preimage of $\bar{f}$ then

$$f \in H^0(\mathcal{I}_{X\cup P^{d-k}}(d)) = (I_X \cap P^{d-k})_d \subset \tilde{K}_k(I_X)_d \text{ for all } d \gg 0.$$

Then f is a homogeneous polynomial of degree d, which is not vanishing on L. This completes the proof. □

As a Corollary, we recover Green's result in [9], which shows multiple locus of projections are defined by partial elimination ideals set-theoretically.

COROLLARY 3.5 ([9]). *Let X be a reduced subscheme of $\mathbb{P}^n$ and let I_X be the homogeneous ideal of X. Let*

$$\pi : \mathbb{P}^n \to \mathbb{P}^{n-1}$$

be an outer projection from the point $p = [1 : 0 : \cdots : 0]$. Set theoretically, the m-th partial elimination ideal $K_m(I_X)$ is the ideal of $\{q \in \pi(X) \mid \text{length}(\pi^{-1}(q)) > m\}$.

Proof. Let $Y_m = \{q \in \pi(Z) \mid \text{length}(\pi^{-1}(q)) > m\}$. Then it is enough to show that

$$Y_m = Z\big(K_m(I_X)\big)$$

($\subset$): For a point $q = [0, a_1, \ldots, a_n] \in Y_m$, if $L = \overline{pq}$ be the line passing through p and q then we see $L \subset Z\big(\tilde{K}_m(I)\big)$ by Theorem 3.4. Let $q' = [t, a_1, \ldots, a_n]$ be a point in L and let $f = x_0^m \bar{f} + g$ is a polynomial of $\tilde{K}_m(I)$. Since f is vanishing on the line L, we see that

$$f(t, q_1, \ldots, q_n) \equiv 0 \text{ for all } t \in k,$$

as a polynomial on the line L with leading coefficient $\bar{f}(q') \in k$. Hence we conclude that $\bar{f}(q') = 0$ and this proves $q \in Z\big(K_m(I_X)\big)$.
($\supset$): We will give a proof by induction on $m \geq 0$. Suppose that $q \in Z\big(K_k(I_X)\big)$ and let L be the line passing through p and q. For $m = 0$, if $f \in \tilde{K}_0(I_X)$ then f can be regarded as a polynomial in $K_0(I_X)$. Since $q \in Z\big(K_0(I_X)\big)$ and $f(q') = f(q) = 0$ for all $q' \in L$, we see that each polynomial in $\tilde{K}_0(I_X)$ is vanishing on L. Then we have $L \subset Z(\tilde{K}_0(I_X))$ and thus it follows from Theroem 3.4 that length$(L \cap X) > 0$. This proves length$(\pi^{-1}(q)) > 0$.

Now suppose that $m > 0$ and $q \in Z\big(K_m(I_X)\big)$. Since we have

$$q \in Z\big(K_m(I_X)\big) \subset Z\big(K_{m-1}(I)\big),$$

we see $\text{mult}_q(\pi(Z)) = \text{length}(L \cap Z) > m - 1$ by induction on m. Note that we have to show that $\text{mult}_q(\pi(Z)) = \text{length}(L \cap Z) > m$. Now assume that

$$\text{mult}_q(\pi(Z)) = \text{length}(L \cap X) \leq m.$$

Then length$(L \cap Z) = m$ and there is a polynomial

$$f = x_0^m \bar{f} + g \in \tilde{K}_m(I_X), \text{ where } d_0(g) \leq m - 1$$

such that f does not vanishing on L by Theorem 3.4. If we write $q = [a_1, \ldots, a_n]$ then points on the line L can be parametrized by $[t, a_1 \ldots, a_n]$. Note that $\bar{f}$ is a polynomial in $K_m(I_X)$ and $q \in Z\big(K_m(I_X)\big)$. Hence we see

$$f(t, a_1 \ldots, a_n) = t^m \bar{f}(q) + g(t, a_1 \ldots, a_n) = g(t, a_1 \ldots, a_n),$$

is a polynomial of degree $m-1$ in a polynomial ring $k[t]$. However,

$$\mathrm{length}(Z(f)\cap L) \geq \mathrm{length}(X\cap L) = m$$

and this contradicts that f is not vanishing on the line L. Consequently, we prove length $(X\cap L) > m$ as we wished. □

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