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A DECOMPOSITION INTO ATOMS OF TENT SPACES ASSOCIATED WITH GENERAL APPROACH REGIONS

Choon-Serk Suh*

Abstract. We first introduce a space of homogeneous type X, and develop the theory of the tent spaces on the generalized upper half-space $X \times (0, \infty)$. The goal of this paper is to study that every element of the tent spaces $T^p_\Omega(X \times (0, \infty), 0 < p \leq 1$, can be decomposed into elementary particles which are called "atoms."

1. Introduction

The theory of the tent spaces on the upper half-space $\mathbb{R}^{n+1}_+$ was introduced from the work of R. R. Coifman, Y. Meyer and E. M. Stein [1]. In this paper we study the theory of the tent spaces on the generalized upper half-space $X \times (0, \infty)$, where X is a space of homogeneous type.

We begin by introducing the notion of a space of homogeneous type [2]: Let X be a topological space endowed with Borel measure μ. Assume that d is a pseudo-metric on X, that is, a nonnegative function defined on $X \times X$ satisfying

(i) $d(x, x) = 0; d(x, y) > 0$ if $x \neq y$,

(ii) $d(x, y) = d(y, x)$, and

(iii) $d(x, z) \leq K(d(x, y) + d(y, z))$, where K is some fixed constant.

Assume further that

(a) the balls $B(x, \rho) = \{y \in X : d(x, y) < \rho\}$, $\rho > 0$, form a basis of open neighborhoods at $x \in X$,

and that μ satisfies the doubling property:

(b) $0 < \mu(B(x, 2\rho)) \leq A\mu(B(x, \rho)) < \infty$, where A is some fixed constant.

Then we call X a *space of homogeneous type.*

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Property (iii) will be referred to as the "triangle inequality." Note that property (b) implies that for every $C > 0$ there is a constant $A_C < \infty$ such that

$$\mu(B(x, C\rho)) \leq A_C \mu(B(x, \rho))$$

for all $x \in X$ and $\rho > 0$.

Note that the volume of balls will be proportional to a fixed power of the radius. Thus assume there are a $\alpha \in \mathbb{R}$ and constants C_1 and C_2 such that

$$C_1 \rho^\alpha \leq \mu(B(x, \rho)) \leq C_2 \rho^\alpha.$$

We will denote $\mu(B(x, \rho)) \approx \rho^\alpha$ for the simplicity of the notation.

Now consider the space $X \times (0, \infty)$, which is a kind of generalized upper half-space over X. Suppose that there is a given set $\Omega_x \subset X \times (0, \infty)$ for each $x \in X$. Let Ω denote the family $\{\Omega_x\}_{x \in X}$. Thus at each $x \in X$, Ω determines a collection of balls, namely, $\{B(y, t) : (y, t) \in \Omega_x\}$.

For a measurable function f defined on $X \times (0, \infty)$, and real number α, we define an area function $S_{\Omega,\alpha}(f)$ of f, with respect to Ω, as

$$S_{\Omega,\alpha}(f)(x) = \left(\int_{\Omega_x} |f(y, t)|^2 \frac{d\mu(y)dt}{t^{\alpha+1}} \right)^{1/2} \tag{1}$$

for $x \in X$. Throughout this paper we will always assume that Ω is chosen so that $S_{\Omega,\alpha}(f)$ is a measurable function on X, and that $\Omega = \{\Omega_x\}_{x \in X}$ is a symmetric family, that is, if $x \in \Omega_y(t)$, then $y \in \Omega_x(t)$, where $\Omega_x(t) = \{y \in X : (y, t) \in \Omega_x\}$.

For any set $E \subset X$, the *tent* over E, with respect to Ω, is the set

$$\hat{E_\Omega} = (X \times (0, \infty)) \setminus \bigcup_{x \notin E} \Omega_x.$$

The *tent space* T^p_Ω is defined as the space of functions f on $X \times (0, \infty)$, so that $S_{\Omega,\alpha}(f) \in L^p(d\mu)$, $0 < p < \infty$, and set

$$||f||_{T^p_\Omega} = ||S_{\Omega,\alpha}(f)||_{L^p(d\mu)}.$$

For $0 < p \leq 1$, a function a, supported in $\hat{B_\Omega}$ for some ball B in X, is said to be an (Ω, p)-*atom* if

$$\int_{\hat{B_\Omega}} |a(x, t)|^2 \frac{d\mu(x)dt}{t} \leq [\mu(B)]^{1-2/p}.$$

We need the notion of points of density: Let F be a closed subset of X whose complement has finite measure. Let γ be a fixed parameter,

$0 < \gamma < 1$. Then we say that a point $x \in X$ has *global γ-density* with respect to F if

$$\frac{\mu(F \cap B(x,\rho))}{\mu(B(x,\rho))} \geq \gamma$$

for *all* balls $B(x,\rho)$ in X. Observe that if F^* is the set of points of global γ-density with respect to F; then F^* is closed, $F^* \subset F$, and

$$(2) \qquad {}^cF^* = \{x \in X : M(\chi_{{}^cF})(x) > 1-\gamma\},$$

where $\chi_{{}^cF}$ is the characteristic function of the open set ${}^cF$, and M is the Hardy-Littlewood maximal operator on X.

2. Main result

Lemma 2.1. *The Hardy-Littlewood maximal operator M is of weak type $(1,1)$. More precisely, if $f \in L^1_{loc}(d\mu)$, then there is a constant C so that*

$$\mu(\{x \in X : M(f)(x) > \lambda\}) \leq C||f||_1/\lambda$$

for all $\lambda > 0$.

Lemma 2.2. *Assume F is a closed subset of X. Then there is a constant C such that*

$$\mu({}^cF^*) \leq C\mu({}^cF),$$

where F^* is the set of points of global γ-density with respect to F.

Proof. Since the Hardy-Littlewood maximal operator M is of weak type (1,1) by Lemma 1, there is a constant C_γ so that

$$(3) \qquad \mu(\{x \in X : M(\chi_{{}^cF})(x) > 1-\gamma\}) \leq C_\gamma||\chi_{{}^cF}||_1/1-\gamma.$$

But the left side of (3) is equal to $\mu({}^cF^*)$ by (2) and so the proof is complete. □

Lemma 2.3. *There are constants C_γ and γ, $0 < \gamma < 1$, sufficiently close to 1, so that whenever F is a closed subset of X whose complement has finite measure and Φ is a nonnegative measurable function defined on $X \times (0,\infty)$, then*

$$\int_{\cup_{x \in F^*}\Omega_x} \Phi(y,t)t^\alpha d\mu(y)dt \leq C_\gamma \int_F \left(\int_{\Omega_x} \Phi(y,t)d\mu(y)dt\right) d\mu(x),$$

where α is given as in (1), and F^* is the set of points of global γ-density with respect to F.

Proof. Observe that Fubini's theorem gives

$$\begin{aligned}&\int_F \left(\int_{\Omega_x} \Phi(y,t) d\mu(y) dt \right) d\mu(x) \\ &\quad = \int_X \Phi(y,t) \left(\int_F \chi_{B(y,t)}(x) d\mu(x) \right) d\mu(y) dt,\end{aligned}$$

where $\chi_{B(y,t)}$ is the characteristic function of the ball $B(y,t)$. Thus it will suffice to show that if

$$(y,t) \in \bigcup_{x \in F^*} \Omega_x,$$

then there is a constant C_γ so that

$$\int_F \chi_{B(y,t)}(x) d\mu(x) \geq C_\gamma t^\alpha. \tag{4}$$

Let

$$(y,t) \in \bigcup_{x \in F^*} \Omega_x.$$

Then there is a point $x \in F^*$ so that $d(x,y) < t$. Now it is obvious by geometric observation that

$$\mu(B(x,t) \cap {}^c B(y,t)) \leq C \mu(B(x,t)), \tag{5}$$

where $C < 1$. However, it is true that

$$\begin{aligned}&\mu(F \cap B(y,t)) + \mu(B(x,t) \cap {}^c B(y,t)) \\ &\quad \geq \mu(F \cap B(x,t) \cap B(y,t)) + \mu(F \cap B(x,t) \cap {}^c B(y,t)) \\ &\quad = \mu(F \cap B(x,t)).\end{aligned} \tag{6}$$

By the global γ-density property, we have

$$\mu(F \cap B(x,t)) \geq \gamma \mu(B(x,t)). \tag{7}$$

Thus (5), (6) and (7) imply that

$$\begin{aligned}&\mu(F \cap B(y,t)) \\ &\quad \geq \mu(F \cap B(x,t)) - \mu(B(x,t) \cap {}^c B(y,t)) \\ &\quad \geq (\gamma - C) \mu(B(x,t)) \\ &\quad = C_\gamma \mu(B(x,t)),\end{aligned}$$

and so, if γ is chosen sufficiently close to 1, then we have

$$\int_F \chi_{B(y,t)}(x) d\mu(x) \geq C_\gamma t^\alpha,$$

since $\mu(B(x,t)) \approx t^{\alpha}$. Thus we get (4). The proof is therefore complete. $\square$

The next lemma is of the type due to Whitney.

LEMMA 2.4. *Let O be an open subset of X. Then there are positive constants $A, h_1 > 1, h_2 > 1$ and $h_3 < 1$ which depend only on the space X, and a sequence $\{B(x_i, \rho_i)\}$ of balls such that*
(i) $\cup_i B(x_i, \rho_i) = O$,
(ii) $B(x_i, h_2\rho_i) \subset O$ and $B(x_i, h_1\rho_i) \cap (X \setminus O) \neq \emptyset$,
(iii) the balls $B(x_i, h_3\rho_i)$ are pairwise disjoint, and
(iv) no point in O lies in more than A of the balls $B(x_i, h_2\rho_i)$.

As the main result of this paper, the following theorem means that every element of the tent spaces $T^p_{\Omega}, 0 < p \leq 1$, can be decomposed into elementary particles which are called "atoms."

THEOREM 2.5. *Let a function f belong to the tent spaces T^p_{Ω}, $0 < p \leq 1$. Then*

$$|f(x,t)| \leq \sum_{j=0}^{\infty} \lambda_j a_j(x,t),$$

where the a_j's are (Ω, p)-atoms, and the λ_j's are positive numbers. Moreover,

$$\sum_{j=0}^{\infty} \lambda_j^p \leq C||S_{\Omega,\alpha}(f)||^p_{L^p(d\mu)}$$

for some constant C.

Proof. For each integer k, let O_k be the open set

$$O_k = {}^cF_k = \{x \in X : S_{\Omega,\alpha}(f)(x) > 2^k\}.$$

Let $O^*_k = {}^cF^*_k$. Then it follows from the notion of global γ-density (with γ sufficiently close to 1) that

$$O^*_k = \{x \in X : M(\chi_{O_k})(x) > 1 - \gamma\}.$$

Observe that for each integer k,

$$O_k \supset O_{k+1},$$
$$O^*_k \supset O_k,$$

and

$$\hat{O}^*_k \supset \hat{O}_k.$$

Moreover, $\cup_{k=-\infty}^{\infty} \hat{O}^*_k$ contains the support of f in $X \times (0, \infty)$. We distinguish two cases:

Case 1. For every integer k, $O_k^* \neq X$. Let

$$O_k^* = \bigcup_{j=0}^{\infty} B_{k,j}$$

be a Whitney decomposition of the open set O_k^*, where

$$B_{k,j} = B(x_{k,j}, \rho_{k,j}).$$

Let

$$\tilde{B}_{k,j} = B(x_{k,j}, Ch_1\rho_{k,j}),$$

where h_1 is given in (ii) of Lemma 4, and C will be chosen sufficiently large in a moment. If $(x,t) \in \hat{O}_k^*$, then $B(x,t) \subset O_k^*$, and $x \in B_{k,j}$ for some j. Let

$$y \in B(x_{k,j}, h_1\rho_{k,j}) \cap (X \setminus O_k^*).$$

Then we have

$$\begin{aligned} t &\leq d(x,y) \\ &\leq K(d(x,x_{k,j}) + d(x_{k,j},y)) \\ &\leq K(1+h_1)\rho_{k,j}, \end{aligned} \tag{8}$$

where K is the constant in the triangle inequality. Hence if $z \in B(x,t)$, then it follows from (8) that

$$\begin{aligned} d(x_{k,j}, z) &\leq K(d(x_{k,j},x) + d(x,z)) \\ &\leq K(\rho_{k,j} + t) \\ &\leq K(\rho_{k,j} + K(1+h_1)\rho_{k,j}) \\ &= K(1 + K(1+h_1))\rho_{k,j}. \end{aligned}$$

Thus if we choose C so that

$$K(1 + K(1+h_1)) < Ch_1,$$

then it follows that

$$B(x,t) \subset B(x_{k,j}, Ch_1\rho_{k,j}),$$

and hence

$$(x,t) \in \widehat{\chi_{\tilde{B}_{k,j}}}$$

Thus we have

$$\hat{O_k^*} \setminus \hat{O_{k+1}^*} = \bigcup_j \Delta_{k,j},$$

where

$$\Delta_{k,j} = \widehat{\chi_{\tilde{B}_{k,j}}} \cap (\hat{O_k^*} \setminus \hat{O_{k+1}^*}).$$

If we let $\chi_{k,j}$ be the characteristic function of the set $\Delta_{k,j}$, then

$$\begin{aligned}|f(y,t)| &\le \sum_{k,j} |f(y,t)|\chi_{k,j}(y,t)\\ &\equiv \sum_{k,j} \lambda_{k,j} a_{k,j}(y,t),\end{aligned}$$

where

$$\begin{aligned}&a_{k,j}(y,t)\\ &\quad = \mu(\tilde{B}_{k,j})^{1/2-1/p}|f(y,t)|\chi_{k,j}(y,t)\left(\int_{\Delta_{k,j}} |f(y,t)|^2 \frac{d\mu(y)dt}{t}\right)^{-1/2},\end{aligned}$$

and

$$\lambda_{k,j} = \mu(\tilde{B}_{k,j})^{-1/2+1/p}\left(\int_{\Delta_{k,j}} |f(y,t)|^2 \frac{d\mu(y)dt}{t}\right)^{1/2}.$$

Now $a_{k,j}$ is an (Ω, p)-atom associated to the ball $\tilde{B}_{k,j}$ since $|f(y,t)| \le 2^{k+1}$ in $(X \times (0,\infty)) \setminus \hat{O^*_{k+1}}$. Also, put

$$\begin{aligned}F &= {}^cO_{k+1},\\ \bigcup_{x\in F^*} \Omega_x &= \hat{O^*_{k+1}},\\ F^* &= {}^cO^*_{k+1},\end{aligned}$$

and

$$\Phi(y,t) = |f(y,t)|^2 \frac{1}{t^{\alpha+1}} \widehat{\chi_{\widetilde{B}_{k,j}}}(y,t),$$

and apply Lemma 3 to get that

$$\begin{aligned}&\int_{\Delta_{k,j}} |f(y,t)|^2 \frac{d\mu(y)dt}{t}\\ &\le \int_{\widehat{\chi_{\widetilde{B}_{k,j}}}\setminus \hat{O^*_{k+1}}} |f(y,t)|^2 \frac{d\mu(y)dt}{t}\\ &\le \int_{{}^c\hat{O^*_{k+1}}} \widehat{\chi_{\widetilde{B}_{k,j}}}(y,t)|f(y,t)|^2 \frac{d\mu(y)dt}{t}\\ &\le C_\gamma \int_{{}^cO_{k+1}} \int_{\Omega_x} |f(y,t)|^2 \widehat{\chi_{\widetilde{B}_{k,j}}}(y,t) \frac{d\mu(y)dt}{t^{\sigma+1}} d\mu(x)\end{aligned}$$

$$\leq C_\gamma \int_{{}^cO_{k+1}\cap\tilde{B}_{k,j}} (S_{\Omega,\alpha}(f)(x))^2 d\mu(x)$$
$$\leq C_\gamma (2^{k+1})^2 \mu(\tilde{B}_{k,j}).$$

Thus we have

$$\begin{aligned}
\sum_{k,j} \lambda_{k,j}^p &= \sum_{k,j} \mu(\tilde{B}_{k,j})^{1-p/2} \left(\int_{\Delta_{k,j}} |f(y,t)|^2 \frac{d\mu(y)dt}{t} \right)^{p/2} \\
&\leq C \sum_{k,j} 2^{pk} \mu(\tilde{B}_{k,j})^{1-p/2} \mu(\tilde{B}_{k,j})^{p/2} \\
&\leq C \sum_{k,j} 2^{pk} \mu(B_{k,j}) \qquad \text{(by the doubling property)} \\
&\leq C \sum_{k} 2^{pk} \mu(O_k^*) \qquad \text{(by Lemma 4)} \\
&\leq C \sum_{k} 2^{pk} \mu(O_k) \qquad \text{(by Lemma 2)} \\
&\leq C ||S_{\Omega,\alpha}(f)||_{L^p(d\mu)}^p.
\end{aligned}$$

Case 2. $O_k^* = X$ for some integer k. Since $||S_{\Omega,\alpha}(f)||_{L^p(d\mu)} < \infty$, there is an integer n so that $O_k^* = X$ for $k \leq n$, and $O_k^* \neq X$ for $k > n$. For $k = n$, let

$$\Delta_n = (X \times (0,\infty)) \setminus \hat{O_{n+1}^*},$$

$$\lambda_n = \mu(X)^{-1/2+1/p} \left(\int_{\Delta_n} |f(y,t)|^2 \frac{d\mu(y)dt}{t} \right)^{1/2},$$

and

$$\begin{aligned}
&a_n(y,t) \\
&= \mu(X)^{-1/p+1/2} |f(y,t)| \chi_{\Delta_n}(y,t) \left(\int_{\Delta_n} |f(y,t)|^2 \frac{d\mu(y)dt}{t} \right)^{-1/2}.
\end{aligned}$$

Then a_n is an (Ω, p)-atom since $|f(y,t)| \leq 2^{n+1}$ in $(X \times (0,\infty)) \setminus \hat{O_{n+1}^*}$. For $k > n$, define $\chi_{k,j}, \lambda_{k,j}$, and $a_{k,j}$ as before. Then we have

$$\begin{aligned}
|f(y,t)| &\leq |f(y,t)| \chi_{\Delta_n}(y,t) + \sum_{k>n,j} |f(y,t)| \chi_{k,j}(y,t) \\
&= \lambda_n a_n(y,t) + \sum_{k>n,j} \lambda_{k,j} a_{k,j}(y,t).
\end{aligned}$$

Finally we have

$$\begin{aligned}
\lambda_n^p = &\ \mu(X)^{-p/2+1}\left(\int_{\Delta_n}|f(y,t)|^2\frac{d\mu(y)dt}{t}\right)^{p/2} \\
\leq &\ C\mu(X)^{-p/2+1}\left(\int_{^cO_{n+1}}\int_{\Omega_x}|f(y,t)|^2\frac{d\mu(y)dt}{t^{\alpha+1}}d\mu(x)\right)^{p/2} \\
\leq &\ C\mu(X)^{-p/2+1}\left(\int_{^cO_{n+1}}(S_{\Omega,\alpha}(f)(x))^2 d\mu(x)\right)^{p/2} \\
\leq &\ C\mu(X) \\
\leq &\ C\mu(O_n) \qquad \text{(by Lemma 2)} \\
\leq &\ C||S_{\Omega,\alpha}(f)||^p_{L^p(d\mu)} \qquad \text{(by the Chebycheff's inequality).}
\end{aligned}$$

Thus, for $k > n$, we have as before

$$\sum_{k,j}\lambda^p_{k,j} \leq C||S_{\Omega,\alpha}(f)||^p_{L^p(d\mu)},$$

and the proof is complete. □

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School of Information and Communications Engineering
Dongyang University
Yeongju 750-711, Republic of Korea
E-mail: cssuh@dyu.ac.kr