

## ON THE HYERS-ULAM-RASSIAS STABILITY OF A BI-PEXIDER FUNCTIONAL EQUATION

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ABSTRACT. In this paper, we study the Hyers-Ulam-Rassias stability of a bi-Pexider functional equation

$$\begin{aligned}f(x + y, z) - f_1(x, z) - f_2(y, z) &= 0, \\f(x, y + z) - f_3(x, y) - f_4(x, z) &= 0.\end{aligned}$$

Moreover, we establish stability results on the punctured domain.

### 1. Introduction

The stability problem of functional equations originated from a question of S. M. Ulam [17] concerning the stability of group homomorphisms: Given a group  $G_1$ , a metric group  $(G_2, d)$  and  $\varepsilon > 0$ , does there exist a  $\delta > 0$  such that if  $h : G_1 \rightarrow G_2$  satisfies

$$d(h(xy), h(x)h(y)) < \delta$$

for all  $x, y \in G_1$ , then a homomorphism  $H : G_1 \rightarrow G_2$  exists with

$$d(h(x), H(x)) < \varepsilon$$

for all  $x \in G_1$ ? If the answer is affirmative, we would say the equation of homomorphism  $H(xy) = H(x)H(y)$  stable.

In 1941, D. H. Hyers [4] gave first affirmative answer to the question of Ulam for Banach spaces. Hyers' theorem was generalized by T. Aoki [1] for additive mappings and by Th. M. Rassias [16] for linear mappings by considering an unbounded Cauchy difference (See the recent Maligranda's paper [13]). Since then, further generalizations of the Hyers-Ulam theorem have been extensively investigated by a number

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of mathematicians [3, 5, 9, 11, 12, 14].

Throughout this paper, let  $X$  be a vector space and  $Y$  a Banach space. A mapping  $g : X \rightarrow Y$  is called a Cauchy mapping (respectively, a Jensen mapping) if  $g$  satisfies the functional equation  $g(x + y) = g(x) + g(y)$  (respectively,  $2g(\frac{x+y}{2}) = g(x) + g(y)$ ).

For given mappings  $f, f_1, f_2, f_3, f_4 : X \times X \rightarrow Y$ , we define

$$\begin{aligned} C_1 f(x, y, z) &:= f(x + y, z) - f(x, z) - f(y, z), \\ C_2 f(x, y, z) &:= f(x, y + z) - f(x, y) - f(x, z), \\ J_1 f(x, y, z) &:= 2f\left(\frac{x+y}{2}, z\right) - f(x, z) - f(y, z), \\ J_2 f(x, y, z) &:= 2f\left(x, \frac{y+z}{2}\right) - f(x, y) - f(x, z), \\ P_1(f, f_1, f_2)(x, y, z) &:= f(x + y, z) - f_1(x, z) - f_2(y, z), \\ P_2(f, f_3, f_4)(x, y, z) &:= f(x, y + z) - f_3(x, y) - f_4(x, z), \\ P(f, f_1, f_2, f_3, f_4)(x, y, z, w) &:= f(x + y, z + w) - f_1(x, z) - f_2(x, w) - f_3(y, z) - f_4(y, w) \end{aligned}$$

for all  $x, y, z, w \in X$ . If a mapping  $f$  satisfies the functional equations  $C_1 f = 0$  and  $C_2 f = 0$  ( $C_1 f = 0$  and  $J_2 f = 0$ ,  $C_2 f = 0$  and  $J_1 f = 0$ ,  $J_1 f = 0$  and  $J_2 f = 0$ ,  $P_1(f, f_1, f_2) = 0$  and  $P_2(f, f_3, f_4) = 0$ , respectively), we say that  $f : X \times X \rightarrow Y$  satisfies a biadditive (Cauchy-Jensen, Jensen-Cauchy, bi-Jensen, bi-Pexider, respectively) functional equation. It is easy to see that  $f$  satisfies a biadditive (Cauchy-Jensen and Jensen-Cauchy respectively) functional equation, then  $f$  satisfies a bi-Jensen functional equation.

In 2006, Bae and Park [2, 15] obtained the generalized Hyers-Ulam stability of a Cauchy-Jensen functional equation and a bi-Jensen functional equation. In 2007, Lee et al. [6, 7, 8, 10] improved the Bae and Park's results.

In this paper, we investigate the stability of a bi-Pexider functional equation.

## 2. Stability of a bi-Jensen functional equation

Throughout in this paper, denote  $P_1(f, f_1, f_2)$  and  $P_2(f, f_3, f_4)$  by  $P_1$  and  $P_2$  briefly, respectively. One can easily prove the basic properties of a bi-Jensen mapping in the following lemmas.

LEMMA 2.1. [7] *Let  $f : X \times X \rightarrow Y$  be a bi-Jensen mapping. Then*

$$\begin{aligned} f(x, y) &= \frac{f(2^n x, 2^n y)}{4^n} + \left(\frac{1}{2^n} - \frac{1}{4^n}\right)(f(2^n x, 0) + f(0, 2^n y)) \\ &\quad + \left(1 - \frac{1}{2^n}\right)^2 f(0, 0), \\ f(x, y) &= \frac{f(2^n x, 2^n y)}{4^n} + (2^n - 1)\left(f\left(\frac{x}{2^n}, 0\right) + f\left(0, \frac{y}{2^n}\right)\right) \\ &\quad - \left(2^{n+1} - 3 + \frac{1}{4^n}\right)f(0, 0), \\ f(x, y) &= 4^n f\left(\frac{x}{2^n}, \frac{y}{2^n}\right) + (2^n - 4^n)\left(f\left(\frac{x}{2^n}, 0\right) + f\left(0, \frac{y}{2^n}\right)\right) \\ &\quad + (2^n - 1)^2 f(0, 0), \\ f(x, y) &= \frac{1}{2^n}(f(2^n x, y) - f(0, y)) + \frac{1}{2^n}(f(0, 2^n y) - f(0, 0)) + f(0, 0) \end{aligned}$$

for all  $x, y \in X$  and  $n \in \mathbb{N}$ .

LEMMA 2.2. *For given mappings  $f, f_1, f_2, f_3, f_4 : X \times X \rightarrow Y$ , let  $f', f'', f''', A_3 : X \times X \rightarrow Y$  and  $A_1, A_2 : X \times X \times X \rightarrow Y$  be defined by*

$$\begin{aligned} f'(x, y) &= f(x, y) - f(0, y), \\ f''(x, y) &= f(x, y) - f(x, 0), \\ f'''(x, y) &= f(x, y) - f(x, 0) - f(0, y) + f(0, 0), \\ A_1(x, y, z) &= P_1(x, y, z) + P_1(y, x, z) - P_1(x, x, z) - P_1(y, y, z), \\ A_2(x, y, z) &= P_2(x, y, z) + P_2(x, z, y) - P_2(x, y, y) - P_2(x, z, z), \\ A_3(x, y) &= \frac{1}{8}(A_1(x, 0, 2y) + 2A_1(x, 0, y) - 3A_1(x, 0, 0) \\ &\quad + A_2(2x, 0, y) + 2A_2(x, 0, y) - 3A_2(0, 0, y)) \end{aligned}$$

for all  $x, y \in X$ . Then

$$\begin{aligned} f'(x, y) - \frac{f'(2x, y)}{2} &= \frac{1}{2}A_1(x, 0, y), \\ f''(x, y) - \frac{f''(x, 2y)}{2} &= \frac{1}{2}A_2(x, 0, y), \\ f'''(x, y) - \frac{f'''(2x, 2y)}{4} &= A_3(x, y), \\ J_1 f'''(x, y, z) &= A_1\left(\frac{x}{2}, \frac{y}{2}, z\right) - A_1\left(\frac{x}{2}, \frac{y}{2}, 0\right), \\ J_2 f'''(x, y, z) &= A_2\left(x, \frac{y}{2}, \frac{z}{2}\right) - A_2\left(0, \frac{y}{2}, \frac{z}{2}\right) \end{aligned}$$

for all  $x, y \in X$ .

**THEOREM 2.3.** *Let  $0 \leq p < 1, 0 < \varepsilon$  and let  $f, f_1, f_2, f_3, f_4 : X \times X \rightarrow Y$  be the mappings such that*

$$(2.1) \quad \|P_1(f, f_1, f_2)(x, y, z)\| \leq \varepsilon(\|x\|^p + \|y\|^p + \|z\|^p),$$

$$(2.2) \quad \|P_2(f, f_3, f_4)(x, y, z)\| \leq \varepsilon(\|x\|^p + \|y\|^p + \|z\|^p)$$

for all  $x, y, z \in X$ . Then there exists a unique bi-Jensen mapping  $F : X \times X \rightarrow Y$  such that

$$(2.3) \quad \|f(x, y) - F(x, y)\| \leq \frac{4\varepsilon}{2 - 2^p} \|x\|^p + \left(\frac{4}{2 - 2^p} + 4\right) \varepsilon \|y\|^p$$

for all  $x, y \in X$  with  $F(0, 0) = f(0, 0)$ . The mapping  $F : X \times X \rightarrow Y$  is given by

$$F(x, y) := \lim_{j \rightarrow \infty} \frac{f(2^j x, y) + f(0, 2^j y)}{2^j} + f(0, 0)$$

for all  $x, y \in X$ .

*Proof.* By Lemma 2.2, (2.1) and (2.2), we get

$$\begin{aligned} \left\| \frac{f(2^j x, y) - f(0, y)}{2^j} - \frac{f(2^{j+1} x, y) - f(0, y)}{2^{j+1}} \right\| &= \left\| \frac{A_1 f(2^j x, 0, y)}{2^{j+1}} \right\| \\ &\leq \frac{(\|2^j x\|^p + \|y\|^p) \varepsilon}{2^{j-1}}, \\ \left\| \frac{f(0, 2^j y) - f(0, 0)}{2^j} - \frac{f(0, 2^{j+1} y) - f(0, 0)}{2^{j+1}} \right\| &= \left\| \frac{A_2 f(0, 0, 2^j y)}{2^{j+1}} \right\| \\ &\leq \frac{\|2^j y\|^p \varepsilon}{2^{j-1}} \end{aligned}$$

for all  $x, y \in X$  and  $j \in \mathbb{N}$ . For given integers  $l, m$  ( $0 \leq l < m$ ),

$$\begin{aligned} \left\| \frac{f'(2^l x, y)}{2^l} - \frac{f'(2^m x, y)}{2^m} \right\| &= \left\| \sum_{j=l}^{m-1} \frac{A_1 f(2^j x, 0, y)}{2^{j+1}} \right\| \\ (2.4) \quad &\leq \sum_{j=l}^{m-1} \frac{2^{jp} \|x\|^p + \|y\|^p}{2^{j-1}} \varepsilon, \end{aligned}$$

$$\begin{aligned} \left\| \frac{f'(0, 2^l y)}{2^l} - \frac{f'(0, 2^m y)}{2^m} \right\| &= \left\| \sum_{j=l}^{m-1} \frac{A_2 f(0, 0, 2^j y)}{2^{j+1}} \right\| \\ (2.5) \quad &\leq \sum_{j=l}^{m-1} \frac{2^{jp} \|y\|^p}{2^{j-1}} \varepsilon \end{aligned}$$

for all  $x, y \in X$ . By  $p < 1$ , both the sequences  $\{\frac{1}{2^j}(f(2^j x, y) - f(0, y))\}$  and  $\{\frac{1}{2^j}(f(0, 2^j y) - f(0, 0))\}$  are Cauchy sequences for all  $x, y \in X$ . Since  $Y$  is complete, the sequences  $\{\frac{1}{2^j}(f(2^j x, y) - f(0, y))\}$  and  $\{\frac{1}{2^j}(f(0, 2^j y) - f(0, 0))\}$  converge for all  $x, y \in X$ . Define  $F_1, F_2 : X \times X \rightarrow Y$  by

$$F_1(x, y) := \lim_{j \rightarrow \infty} \frac{f(2^j x, y)}{2^j},$$

$$F_2(x, y) := \lim_{j \rightarrow \infty} \frac{f(0, 2^j y)}{2^j}$$

for all  $x, y \in X$ . Putting  $l = 0$  and taking  $m \rightarrow \infty$  in (2.4) and (2.5), then one can obtain the inequalities

$$\|f(x, y) - f(0, y) - F_1(x, y)\| \leq \frac{4\varepsilon}{2 - 2^p} \|x\|^p + 4\varepsilon \|y\|^p,$$

$$\|f(0, y) - f(0, 0) - F_2(x, y)\| \leq \frac{4\varepsilon}{2 - 2^p} \|y\|^p$$

for all  $x, y \in X$ . By (2.1), (2.2) and the definitions of  $F_1$  and  $F_2$ , we get

$$J_1 F_1(x, y, z) = \lim_{j \rightarrow \infty} \frac{A_1(2^{j-1}x, 2^{j-1}y, z)}{2^j} = 0,$$

$$J_2 F_1(x, y, z) = \lim_{j \rightarrow \infty} \frac{A_2(2^j x, y, z) - A_2(2^j x, \frac{y+z}{2}, \frac{y+z}{2})}{2^j} = 0,$$

$$J_1 F_2(x, y, z) = 0,$$

$$J_2 F_2(x, y, z) = \lim_{j \rightarrow \infty} \frac{A_2(0, 2^{j-1}y, 2^{j-1}z)}{2^j} = 0$$

for all  $x, y, z \in X$  and so  $F$  is a bi-Jensen mapping satisfying (2.3), where  $F$  is given by

$$F(x, y) = F_1(x, y) + F_2(x, y) + f(0, 0).$$

Now, let  $F' : X \times X \rightarrow Y$  be another bi-Jensen mapping satisfying (2.3) with  $F'(0, 0) = f(0, 0)$ . By Lemma 2.1, we have

$$\begin{aligned} \|F(x, y) - F'(x, y)\| &= \frac{1}{2^n} \|(F - F')(2^n x, y) + (1 - \frac{1}{2^n})(F - F')(0, 2^n y)\| \\ &\leq \frac{1}{2^n} (\|(F - f)(2^n x, y)\| + \|(F - f)(0, 2^n y)\|) \end{aligned}$$

$$\begin{aligned}
& + \|(f - F')(2^n x, y)\| + \|(f - F')(0, 2^n y)\| \\
& \leq \left(\frac{2^p}{2}\right)^n \frac{8\varepsilon}{2 - 2^p} \|x\|^p + \left(\frac{2^{np+1}}{2^n}\right) \left(\frac{8}{2 - 2^p} + 8\right) \varepsilon \|y\|^p
\end{aligned}$$

for all  $x, y \in X$  and  $n \in \mathbb{N}$ . As  $n \rightarrow \infty$ , we may conclude that  $F(x, y) = F'(x, y)$  for all  $x, y \in X$ . Thus such a bi-Jensen mapping  $F : X \times X \rightarrow Y$  is unique.  $\square$

Let  $f, f_1, f_2, f_3, f_4, F, F' : X \times X \rightarrow Y$  be the bi-Jensen maps defined by

$$f(x, y) = f_1(x, y) = f_2(x, y) = f_3(x, y) = f_4(x, y) := 0,$$

$$F(x, y) := \varepsilon, F'(x, y) := -\varepsilon$$

for all  $x, y \in X$ . Then  $f, f_1, f_2, f_3, f_4, F, F'$  satisfy the conditions in Theorem 2.3 but  $F' \neq F$ . Hence the condition  $F(0, 0) = f(0, 0)$  is necessary to show that the map  $F$  is unique.

**THEOREM 2.4.** *Let  $2 < p$  and let  $f, f_1, f_2, f_3, f_4 : X \times X \rightarrow Y$  be the mappings satisfying (2.1) and (2.2) for all  $x, y, z \in X$ . Then there exists a unique bi-Jensen mapping  $F : X \times X \rightarrow Y$  such that*

$$(2.6) \quad \|f(x, y) - F(x, y)\| \leq \left(\frac{16}{2^p - 4} + \frac{2 \cdot 2^p}{2^p - 4} + \frac{4}{2^p - 2}\right) \varepsilon (\|x\|^p + \|y\|^p)$$

for all  $x, y \in X$ . The mapping  $F$  is given by

$$\begin{aligned}
F(x, y) := \lim_{j \rightarrow \infty} & \left( 4^j f\left(\frac{x}{2^j}, \frac{y}{2^j}\right) - (4^j - 2^j) \left( f\left(\frac{x}{2^j}, 0\right) + f\left(0, \frac{y}{2^j}\right) \right) \right. \\
& \left. + (2^j - 1)^2 f(0, 0) \right)
\end{aligned}$$

for all  $x, y \in X$ .

*Proof.* By Lemma 2.2, (2.1) and (2.2), we get

$$\begin{aligned}
& \|2^l (f(\frac{x}{2^l}, 0) - f(0, 0)) - 2^m (f(\frac{x}{2^m}, 0) - f(0, 0))\| \\
(2.7) \quad & = \left\| \sum_{j=l}^{m-1} 2^{j+1} A_1\left(\frac{x}{2^{j+1}}, 0, 0\right) \right\| \leq \sum_{j=l}^{m-1} \frac{2^{j+3}}{2^{(j+1)p}} \varepsilon \|x\|^p,
\end{aligned}$$

$$\begin{aligned}
& \|2^l(f(0, \frac{y}{2^l}) - f(0, 0)) - 2^m(f(0, \frac{y}{2^m}) - f(0, 0))\| \\
(2.8) \quad &= \left\| \sum_{j=l}^{m-1} 2^{j+1} A_2(0, 0, \frac{y}{2^{j+1}}) \right\| \leq \sum_{j=l}^{m-1} \frac{2^{j+3}}{2^{(j+1)p}} \varepsilon \|y\|^p,
\end{aligned}$$

$$\begin{aligned}
& \|4^l f'''(\frac{x}{2^l}, \frac{y}{2^l}) - 4^m f'''(\frac{x}{2^m}, \frac{y}{2^m})\| = \left\| \sum_{j=l}^{m-1} 4^{j+1} A_3(\frac{x}{2^{j+1}}, \frac{y}{2^{j+1}}) \right\| \\
(2.9) \quad &\leq \sum_{j=l}^{m-1} \left( \frac{4^{j+2}}{2^{(j+1)p}} + \frac{4^{j+1}}{2 \cdot 2^{jp}} \right) \varepsilon (\|x\|^p + \|y\|^p)
\end{aligned}$$

for all  $x, y \in X$  and given integers  $l, m$  ( $0 \leq l < m$ ). By the similar method in Theorem 2.3, we can define  $F_1, F_2, F_3 : X \times X \rightarrow Y$  by

$$\begin{aligned}
F_1(x, y) &:= \lim_{j \rightarrow \infty} 2^j (f(\frac{x}{2^j}, 0) - f(0, 0)), \\
F_2(x, y) &:= \lim_{j \rightarrow \infty} 2^j (f(0, \frac{y}{2^j}) - f(0, 0)), \\
F_3(x, y) &:= \lim_{j \rightarrow \infty} 4^j f'''(\frac{x}{2^j}, \frac{y}{2^j})
\end{aligned}$$

for all  $x, y \in X$ . Putting  $l = 0$  and taking  $m \rightarrow \infty$  in (2.7), (2.8) and (2.9), one can obtain the inequalities

$$\begin{aligned}
\|f(0, y) - f(0, 0) - F_1(x, y)\| &\leq \frac{4\varepsilon}{2^p - 2} \|x\|^p, \\
\|f(0, y) - f(0, 0) - F_2(x, y)\| &\leq \frac{4\varepsilon}{2^p - 2} \|y\|^p, \\
\|f'''(x, y) - F_3(x, y)\| &\leq \left( \frac{16}{2^p - 4} + \frac{2 \cdot 2^p}{2^p - 4} \right) \varepsilon (\|x\|^p + \|y\|^p)
\end{aligned}$$

By Lemma 2.2, (2.1), (2.2) and the definitions of  $F_1$  and  $F_2$ , we get

$$\begin{aligned}
J_1 F_1(x, y, z) &= \lim_{j \rightarrow \infty} 2^j A_1(\frac{x}{2^{j+1}}, \frac{y}{2^{j+1}}, 0) = 0, \\
J_2 F_1(x, y, z) &= 0, \\
J_1 F_2(x, y, z) &= 0, \\
J_2 F_2(x, y, z) &= \lim_{j \rightarrow \infty} 2^j A_2(0, \frac{y}{2^{j+1}}, \frac{z}{2^{j+1}}) = 0, \\
J_1 F_3(x, y, z) &= \lim_{j \rightarrow \infty} 4^j (A_1(\frac{x}{2^{j+1}}, \frac{y}{2^{j+1}}, \frac{z}{2^j}) - A_1(\frac{x}{2^{j+1}}, \frac{y}{2^{j+1}}, 0)) = 0, \\
J_2 F_3(x, y, z) &= \lim_{j \rightarrow \infty} 4^j (A_2(\frac{x}{2^j}, \frac{y}{2^{j+1}}, \frac{z}{2^{j+1}}) - A_2(0, \frac{y}{2^{j+1}}, \frac{z}{2^{j+1}})) = 0
\end{aligned}$$

for all  $x, y, z \in X$  and so  $F$  is a bi-Jensen mapping satisfying (2.6) where  $F$  is given by

$$F(x, y) = F_1(x, y) + F_2(x, y) + F_3(x, y) + f(0, 0).$$

Now, let  $F' : X \times X \rightarrow Y$  be another bi-Jensen mapping satisfying (2.6) with  $F'(0, 0) = f(0, 0)$ . By Lemma 2.1 and  $F'(0, 0) = f(0, 0) = F(0, 0)$ , we have

$$\begin{aligned} & \|F(x, y) - F'(x, y)\| \\ &= \|4^n(F - F')(\frac{x}{2^n}, \frac{y}{2^n}) + (2^n - 4^n)((F - F')(\frac{x}{2^n}, 0) \\ &\quad + (F - F')(0, \frac{y}{2^n}))\| \\ &\leq 4^n(\|(F - f)(\frac{x}{2^n}, \frac{y}{2^n})\| + \|(f - F')(\frac{x}{2^n}, \frac{y}{2^n})\| + \|(F - f)(\frac{x}{2^n}, 0)\| \\ &\quad + \|(f - F')(0, \frac{y}{2^n})\| + \|(F - f)(0, \frac{y}{2^n})\| + \|(f - F')(0, \frac{y}{2^n})\|) \\ &\leq \frac{4^{n+1}}{2^{np}}(\frac{16}{2^p - 4} + \frac{2 \cdot 2^p}{2^p - 4} + \frac{4}{2^p - 2})\varepsilon(\|x\|^p + \|y\|^p) \end{aligned}$$

for all  $x, y \in X$  and  $n \in \mathbb{N}$ . As  $n \rightarrow \infty$ , we may conclude that  $F(x, y) = F'(x, y)$  for all  $x, y \in X$ . Thus such a bi-Jensen mapping  $F : X \times X \rightarrow Y$  is unique.  $\square$

**THEOREM 2.5.** *Let  $1 < p < 2$  and let  $f, f_1, f_2, f_3, f_4 : X \times X \rightarrow Y$  be the mappings satisfying (2.1) and (2.2) for all  $x, y, z \in X$ . Then there exists a unique bi-Jensen mapping  $F : X \times X \rightarrow Y$  such that*

$$(2.10) \quad \|f(x, y) - F(x, y)\| \leq (\frac{16}{4 - 2^p} + \frac{2 \cdot 2^p}{4 - 2^p} + \frac{4}{2^p - 2})\varepsilon(\|x\|^p + \|y\|^p)$$

for all  $x, y \in X$ . The mapping  $F$  is given by

$$\begin{aligned} F(x, y) &:= \lim_{j \rightarrow \infty} \frac{1}{4^j} [f(2^j x, 2^j y) - f(2^j x, 0) - f(0, 2^j y)] \\ &\quad + \lim_{j \rightarrow \infty} [2^j(f(\frac{x}{2^j}, 0) + f(0, \frac{y}{2^j})) - (2^{j+1} - 1)f(0, 0)] \end{aligned}$$

for all  $x, y \in X$ .

*Proof.* Let  $F_1, F_2$  be as in the proof of Theorem 2.4. By Lemma 2.2, (2.1) and (2.2), we get

$$\begin{aligned} & \|\frac{1}{4^j} f'''(2^j x, 2^j y) - \frac{1}{4^{j+1}} f'''(2^{j+1} x, 2^{j+1} y)\| = \frac{1}{4^j} \|A_3(2^j x, 2^j y)\| \\ & \leq (\frac{2^{jp}}{4^{j-1}} + \frac{2^{(j+1)p}}{2 \cdot 4^j})\varepsilon(\|x\|^p + \|y\|^p) \end{aligned}$$



for all  $x, y \in X$  and  $j \in \mathbb{N}$ . By the similar method in Theorem 2.3, we define  $F_3 : X \times X \rightarrow Y$  by

$$F_3(x, y) := \lim_{j \rightarrow \infty} \frac{1}{4^j} f'''(2^j x, 2^j y)$$

for all  $x, y \in X$  and obtain the inequality

$$\|f'''(x, y) - F_3(x, y)\| \leq \left(\frac{16}{4 - 2^p} + \frac{2 \cdot 2^p}{4 - 2^p}\right)(\|x\|^p + \|y\|^p)$$

for all  $x, y \in X$ . By (2.1), (2.2) and the definition of  $F_3$ , we get

$$\begin{aligned} J_1 F_3(x, y, z) &= \lim_{j \rightarrow \infty} \frac{A_1(2^{j-1}x, 2^{j-1}y, 2^j z) - A_1(2^{j-1}x, 2^{j-1}y, 0)}{4^j} = 0, \\ J_2 F_3(x, y, z) &= \lim_{j \rightarrow \infty} \frac{A_2(2^j x, 2^{j-1}y, 2^{j-1}z) - A_2(0, 2^{j-1}y, 2^{j-1}z)}{4^j} = 0 \end{aligned}$$

for all  $x, y, z \in X$  and so  $F$  is a bi-Jensen mapping satisfying (2.10) where  $F$  is given by

$$F(x, y) = F_1(x, y) + F_2(x, y) + F_3(x, y) + f(0, 0).$$

Now, let  $F' : X \times X \rightarrow Y$  be another bi-Jensen mapping satisfying (2.10) with  $F'(0, 0) = f(0, 0)$ . By Lemma 2.1 and  $F'(0, 0) = f(0, 0) = F(0, 0)$ , we have

$$\begin{aligned} &\|F(x, y) - F'(x, y)\| \\ &= \left\| \frac{(F - F')(2^n x, 2^n y)}{4^n} + (2^n - 1) \left( (F - F')\left(\frac{x}{2^n}, 0\right) \right. \right. \\ &\quad \left. \left. + (F - F')\left(0, \frac{y}{2^n}\right) \right) \right\| \\ &\leq \left\| \frac{(F - f)(2^n x, 2^n y)}{4^n} \right\| + \left\| \frac{(f - F')(2^n x, 2^n y)}{4^n} \right\| + 2^n (\|(F - f)\left(\frac{x}{2^n}, 0\right)\| \\ &\quad + 2^n \|(F - f)\left(0, \frac{y}{2^n}\right)\| + \|(f - F')\left(\frac{x}{2^n}, 0\right)\| + \|(f - F')\left(0, \frac{y}{2^n}\right)\|) \\ &\leq \left(\frac{2^{np}}{4^n} + \frac{2^n}{2^{np}}\right) \left(\frac{32}{4 - 2^p} + \frac{4 \cdot 2^p}{4 - 2^p} + \frac{8}{2^p - 2}\right) \varepsilon(\|x\|^p + \|y\|^p) \end{aligned}$$

for all  $x, y \in X$  and  $n \in \mathbb{N}$ . As  $n \rightarrow \infty$ , we may conclude that  $F(x, y) = F'(x, y)$  for all  $x, y \in X$ . Thus such a bi-Jensen mapping  $F : X \times X \rightarrow Y$  is unique.  $\square$

THEOREM 2.6. Let  $0 \leq p(\neq 1), 0 < \varepsilon$  and let  $f, f_1, f_2, f_3, f_4 : X \times X \rightarrow Y$  be the mappings such that

$$(2.11) \quad \|P_1(f, f_1, f_2)(x, y, z)\| \leq \varepsilon(\|x\|^p + \|y\|^p)\|z\|^p,$$

$$(2.12) \quad \|P_2(f, f_3, f_4)(x, y, z)\| \leq \varepsilon\|x\|^p(\|y\|^p + \|z\|^p)$$

for all  $x, y, z \in X$ . Then there exists a unique bi-Jensen mapping  $F : X \times X \rightarrow Y$  such that

$$\|f(x, y) - F(x, y)\| \leq \frac{4\varepsilon}{|2 - 2^p|} \|x\|^p \|y\|^p$$

for all  $x, y \in X$  with  $F(0, 0) = f(0, 0)$ . The mapping  $F : X \times X \rightarrow Y$  is given by

$$F(x, y) = \lim_{j \rightarrow \infty} \frac{f(2^j x, y)}{2^j} + f(0, y) \quad \text{if } 0 \leq p < 1,$$

$$F(x, y) = \lim_{j \rightarrow \infty} 2^j (f(\frac{x}{2^j}, y) - f(0, y)) + f(0, y) \quad \text{if } p > 1.$$

*Proof.* By Lemma 2.2, (2.11) and (2.12), we get

$$\begin{aligned} \left\| \frac{f(2^j x, y) - f(0, y)}{2^{j+1}} - \frac{f(2^{j+1} x, y) - f(0, y)}{2^{j+1}} \right\| &= \left\| \frac{A_1(2^j x, 0, y)}{2^{j+1}} \right\| \\ &\leq \frac{2^{jp} \|x\|^p \|y\|^p}{2^{j-1}} \varepsilon, \\ \|f(0, y) - f(0, 0) - \frac{f(0, 2^n y) - f(0, 0)}{2^n}\| &= \left\| \sum_{j=0}^{n-1} \frac{A_2(0, 0, 2^j y)}{2^{j+1}} \right\| = 0 \end{aligned}$$

for all  $x, y \in X, j \in \mathbb{N}$  if  $0 \leq p < 1$  and

$$\begin{aligned} &\|2^j (f(\frac{x}{2^j}, y) - f(0, y)) - 2^{j+1} f(\frac{x}{2^{j+1}}, y) - f(0, y)\| \\ &= \|2^j A_1(\frac{x}{2^{j+1}}, 0, y)\| \leq \frac{2^{j+2} \|x\|^p \|y\|^p}{2^{(j+1)p}} \varepsilon, \\ \|f(0, y) - f(0, 0) - 2^n (f(0, \frac{y}{2^n}) - f(0, 0))\| \\ &= \left\| \sum_{j=0}^{n-1} 2^{j+1} A_2(0, 0, \frac{y}{2^{j+1}}) \right\| = 0 \end{aligned}$$

for all  $x, y \in X, j \in \mathbb{N}$  if  $p > 1$ . The remainder of proof is same to the proof of Theorem 2.3.  $\square$

THEOREM 2.7. Let  $0 \leq p(\neq 1), 0 < \varepsilon$  and let  $f, f_1, f_2, f_3, f_4 : X \times X \rightarrow Y$  be the mappings satisfying (2.11) and (2.12) for all  $x, y, z \in X$ .

Then there exists a unique bi-Jensen mapping  $F : X \times X \rightarrow Y$  such that

$$\|f(x, y) - F(x, y)\| \leq \frac{\varepsilon}{|4 - 4^p|} \|x\|^p \|y\|^p$$

for all  $x, y \in X$ . The mapping  $F : X \times X \rightarrow Y$  is given by

$$\begin{aligned} F(x, y) &= \lim_{j \rightarrow \infty} \frac{f(2^j x, 2^j y)}{4^j} + f(x, 0) + f(y, 0) - f(0, 0) \quad \text{if } 0 \leq p < 1, \\ F(x, y) &= \lim_{j \rightarrow \infty} 4^j \left( f\left(\frac{x}{2^j}, \frac{y}{2^j}\right) - f\left(\frac{x}{2^j}, 0\right) - f\left(0, \frac{y}{2^j}\right) + f(0, 0) \right) \\ &\quad + f(x, 0) + f(y, 0) - f(0, 0) \quad \text{if } p > 1. \end{aligned}$$

*Proof.* By Lemma 2.2, (2.11) and (2.12), we get

$$\begin{aligned} \left\| \frac{f'''(2^j x, 2^j y)}{4^j} - \frac{f'''(2^{j+1} x, 2^{j+1} y)}{4^{j+1}} \right\| &= \left\| \frac{A_3(2^j x, 2^j y)}{4^j} \right\| \\ &\leq \frac{4^{jp}(2 + 2^p)}{4^j} \varepsilon \|x\|^p \|y\|^p \end{aligned}$$

for all  $x, y \in X, j \in \mathbb{N}$  if  $0 \leq p < 1$  and

$$\begin{aligned} \|4^j f'''(\frac{x}{2^j}, \frac{y}{2^j}) - 4^{j+1} f'''(\frac{x}{2^{j+1}}, \frac{y}{2^{j+1}})\| &= 4^{j+1} \|A_3(\frac{x}{2^{j+1}}, \frac{y}{2^{j+1}})\| \\ &\leq \frac{4^{j+1}(2 + 2^p)}{4^{(j+1)p}} \varepsilon \|x\|^p \|y\|^p \end{aligned}$$

for all  $x, y \in X, j \in \mathbb{N}$  if  $p > 1$ . As in the proof of Theorem 2.6, we get

$$\begin{aligned} f(x, 0) - f(0, 0) &= \frac{f(2^n x, 0) - f(0, 0)}{2^n} = 2^n \left( f\left(\frac{x}{2^n}, 0\right) - f(0, 0) \right), \\ f(0, y) - f(0, 0) &= \frac{f(0, 2^n y) - f(0, 0)}{2^n} = 2^n \left( f\left(0, \frac{y}{2^n}\right) - f(0, 0) \right) \end{aligned}$$

for all  $x, y \in X, n \in \mathbb{N}$ . The remainder of proof is same to the proof of Theorem 4.  $\square$

### 3. Stability of a bi-Pexider functional equation on the punctured domain

The following theorem can be found in [8].

**THEOREM 3.1.** Let  $p < 0$  and  $\varepsilon > 0$ . Let  $f : X \times X \rightarrow Y$  be a mapping such that

$$\begin{aligned} \|J_1 f(x, y, z)\| &\leq \varepsilon (\|x\|^p + \|y\|^p + \|z\|^p) \\ \|J_2 f(x, y, z)\| &\leq \varepsilon (\|x\|^p + \|y\|^p + \|z\|^p) \end{aligned}$$

for all  $x, y, z \in X \setminus \{0\}$ . Then there exists a (unique for the bi-Jensen mapping  $F'$  with  $F(0, 0) = F'(0, 0)$ ) bi-Jensen mapping  $F : X \times X \rightarrow Y$  such that

$$\|f(x, y) - F(x, y)\| \leq \frac{3 - 2^p + 3^p}{2 - 2^p} \varepsilon \|x\|^p + \frac{12 - 3 \cdot 2^p + 2 \cdot 3^p}{2(2 - 2^p)} \varepsilon \|y\|^p$$

for all  $x, y \in X \setminus \{0\}$ .

**THEOREM 3.2.** Let  $p < 0$  and  $\varepsilon > 0$  and let  $f, f_1, f_2, f_3, f_4 : X \times X \rightarrow Y$  be the mappings satisfying (2.1) and (2.2) for all  $x, y, z \in X \setminus \{0\}$ . Then there exists a (unique for the bi-Jensen mapping  $F'$  with  $F(0, 0) = F'(0, 0)$ ) bi-Jensen mapping  $F : X \times X \rightarrow Y$  such that

$$\|f(x, y) - F(x, y)\| \leq \frac{12 - 4 \cdot 2^p + 4 \cdot 3^p}{2^p(2 - 2^p)} \varepsilon \|x\|^p + \frac{24 - 6 \cdot 2^p + 4 \cdot 3^p}{2^p(2 - 2^p)} \varepsilon \|y\|^p$$

for all  $x, y \in X \setminus \{0\}$ .

*Proof.* Since

$$\|J_1 f(x, y, z)\| = \|A_1(\frac{x}{2}, \frac{y}{2}, z)\| \leq \frac{4\varepsilon}{2^p} (\|x\|^p + \|y\|^p + \|2z\|^p),$$

$$\|J_2 f(x, y, z)\| = \|A_2(x, \frac{y}{2}, \frac{z}{2})\| \leq \frac{4\varepsilon}{2^p} (\|x\|^p + \|y\|^p + \|2z\|^p)$$

for all  $x, y, 2z \in X \setminus \{0\}$ , we can apply Theorem 3.1 and obtain the desired result.  $\square$

The following theorem can be found in [8].

**THEOREM 3.3.** Let  $p < 0$  and  $\varepsilon > 0$ . Let  $f : X \times X \rightarrow Y$  be a mapping such that

$$\|J_1 f(x, y, z)\| \leq \varepsilon (\|x\|^p + \|y\|^p) \|z\|^p,$$

$$\|J_2 f(x, y, z)\| \leq \varepsilon \|x\|^p (\|y\|^p + \|z\|^p)$$

for all  $x, y, z \in X \setminus \{0\}$ . Then there exists a bi-Jensen mapping  $F : X \times X \rightarrow Y$  such that

$$f(x, y) = F(x, y)$$

for all  $(x, y) \neq (0, 0)$ .

**THEOREM 3.4.** Let  $p < 0$  and  $\varepsilon > 0$ . Let  $f, f_1, f_2, f_3, f_4 : X \times X \rightarrow Y$  be the mappings satisfying (2.11) and (2.12) for all  $x, y, z \in X \setminus \{0\}$ . Then there exists a bi-Jensen mapping  $F : X \times X \rightarrow Y$  such that

$$f(x, y) = F(x, y)$$

for all  $(x, y) \neq (0, 0)$ .

*Proof.* Since

$$\begin{aligned}\|J_1 f(x, y, z)\| &= \|A_1(\frac{x}{2}, \frac{y}{2}, z)\| \leq \frac{4\varepsilon}{2^p}(\|x\|^p + \|y\|^p)\|z\|^p, \\ \|J_2 f(x, y, z)\| &= \|A_2(x, \frac{y}{2}, \frac{z}{2})\| \leq \frac{4\varepsilon}{2^p}\|x\|^p(\|y\|^p + \|z\|^p)\end{aligned}$$

for all  $x, y, z \in X \setminus A$ . Hence we can apply Theorem 3.3 and obtain the desired result.  $\square$

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