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ON FUZZY SUBHYPERNEAR-RINGS OF HYPERNEAR-RINGS WITH t-NORMS

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Abstract. In this paper, we investigate some properties of T-fuzzy subhypernear-rings of a hypernear-ring.

1. Introduction

The theory of hyperstructures has been introduced by Marty in 1934 during the 8^{th} congress of the Scandinavian Mathematicians [16]. Marty introduced the notion of a hypergroup and then many researchers have been worked on this new field of modern algebra and developed it. A comprehensive review of the theory of hyperstructures appear [5] and [20]. The notion of the hyperfield and hyperring was studied by Krasner [14]. In [6], Dasic has introduced the notion of hypernear-rings generalizing the concept of near-ring [17]. In [11], Gontineac defined the zero-symmetric part and the constant part of a hypernear-ring and introduced a structure theorem and other properties of hypernear-rings. Davvaz in [8] introduced the notion of an H_v-near ring generalizing the notion of hypernear-ring.

In [7], Davvaz has introduced the concept of fuzzy subhypernear-rings and fuzzy hyperideals of a hypernear-ring which are a generalization of the concept of a fuzzy subnear-rings and fuzzy ideals in a near-ring. Now, in this paper, we investigate some properties of T-fuzzy subhypernear-rings In this paper, we investigate some properties of T-fuzzy subhypernear-rings of a hypernear-ring.

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2. Preliminaries

Let H be a non-empty set. A *hyperoperation* $*$ on H is a mapping of $H \times H$ into the family of non-empty subsets of H.

A *hypernear-ring* is an algebraic structure $(R, +, \cdot)$ which satisfies the following axioms:

(1) $(R, +)$ is a hypergroup i.e., in $(R, +)$ the following hold:

- (i) $x + (y + z) = (x + y) + z$ for all $x, y, z \in R$;
- (ii) There is $0 \in R$ such that $x + 0 = 0 + x = x$ for all $x \in R$;
- (iii) For every $x \in R$ there exists one and only one $x' \in R$ such that $0 \in x + x'$, (we shall write $-x$ for x' and we call it the opposite of x);
- (iv) $z \in x + y$ implies $y \in -x + z$ and $x \in z - y$.

If $x \in R$ and A, B are subsets of R, then by $A + B$, $A + x$ and $x + B$ we mean

$$A + B = \bigcup_{\substack{a \in A \\ b \in B}} a + b, A + x = A + \{x\} \ , \ x + B = \{x\} + B.$$

(2) With respect to the multiplication, $(R, \cdot)$ is a semigroup having absorbing element 0 i.e., $x \cdot 0 = 0$ for all $x \in R$.

(3) The multiplication is distributive with respect to the hyperoperation $+$ on the left side i.e., $x \cdot (y + z) = x \cdot y + x \cdot z$ for all $x, y, z \in R$.

Note that for all $x, y \in R$, we have $-(-x) = x, 0 = -0, -(x + y) = -y - x$ and $x(-y) = -xy$.

Let $(R, +, \cdot)$ be a hypernear-ring. A non-empty subset H of R is a *subhypernear-ring* if

1. (1) $(H, +)$ is a subhypergroup of $(R, +)$, i.e., $a, b \in H$ implies $a + b \subseteq H$, and $a \in H$ implies $-a \in H$,
2. (2) $ab \in H$ for all $a, b \in H$.

Example 2.1. Consider hypernear-ring $R = \{0, a, b\}$ with two binary operations as follows:

$+$	0	a	b
0	$\{0\}$	$\{a\}$	$\{b\}$
a	$\{a\}$	$\{0, a, b\}$	$\{a, b\}$
b	$\{b\}$	$\{a, b\}$	$\{0, a, b\}$

$\cdot$	0	a	b
0	0	0	0
a	0	a	b
b	0	a	b

Then $(R, +, \cdot)$ is a hypernear-ring and $\{0\}$ and R are subhypernear-rings of R.

A *fuzzy subset* μ in a set R is a function $\mu : R \to [0,1]$ and $\mathrm{Im}(\mu)$ denote the *image set* of μ.

Definition 2.2. Let $(R,+,\cdot)$ be a hypernear-ring and μ a fuzzy subset of R. We say that μ is a *fuzzy subhypernear-ring* of R if

(H1) $\min\{\mu(x),\mu(y)\} \le \inf_{\alpha\in x+y}\{\mu(\alpha)\}$ for all $x,y \in R$,
(H2) $\mu(x) \le \mu(-x)$,
(H3) $\mu(xy) \ge \min\{\mu(x),\mu(y)\}$ for all $x,y \in R$.

Definition 2.3. ([5]) By a *t-norm* T, we mean a function $T : [0,1]\times[0,1] \to [0,1]$ satisfying the following conditions:
(T1) $T(x,1) = x$,
(T2) $T(x,y) \le T(x,z)$ if $y \le z$,
(T3) $T(x,y) = T(y,x)$,
(T4) $T(x,T(y,z)) = T(T(x,y),z)$,
for all $x,y,z \in [0,1]$.

For a t-norm T on $[0,1]$, denote by Δ_T the set of element $\alpha \in [0,1]$ such that $T(\alpha,\alpha)=\alpha$, i.e., $\Delta_T := \{\alpha \in [0,1] \mid T(\alpha,\alpha)=\alpha\}$.

Proposition 2.4. *Every t-norm T has a useful property:*

$$T(\alpha,\beta) \le \min(\alpha,\beta)$$

for all $\alpha,\beta \in [0,1]$.

Definition 2.5. Let T be a t-norm. A fuzzy subset μ of R is said to satisfy *idempotent property* if $\mathrm{Im}(\mu) \subseteq \Delta_T$.

3. Fuzzy subhypernear-rings of hypernear-rings with t-norms

Definition 3.1. Let $(R,+,\cdot)$ be a hypernear-ring and μ a fuzzy subset of R. We say that μ is a *fuzzy subhypernear-ring of R with respect to t-norm T* (briefly, a *T-fuzzy subhypernear-ring of R*) if

(TH1) $T(\mu(x),\mu(y)) \le \inf_{\alpha\in x+y}\{\mu(\alpha)\}$ for all $x,y \in R$,
(TH2) $\mu(x) \le \mu(-x)$,
(TH3) $\mu(xy) \ge T(\mu(x),\mu(y))$ for all $x,y \in R$.

EXAMPLE 3.2. Let $R = \{0, a, b, c\}$ be a set with a hyperoperation "$+$" and a binary operation "$\cdot$" as follows:

$+$	0	a	b	c
0	$\{0\}$	$\{a\}$	$\{b\}$	$\{c\}$
a	$\{a\}$	$\{0,a\}$	$\{b\}$	$\{c\}$
b	$\{b\}$	$\{b\}$	$\{0,a,c\}$	$\{b,c\}$
c	$\{c\}$	$\{c\}$	$\{b,c\}$	$\{0,a,b\}$

$\cdot$	0	a	b	c
0	0	a	b	c
a	0	a	b	c
b	0	a	b	c
c	0	a	b	c

Then $(R, +, \cdot)$ is a hypernear-ring. We define a fuzzy set μ in R by

$$\mu(0) = 0.7, \mu(a) = 0.5 \text{ and } \mu(b) = \mu(c) = 0.3.$$

Let $T : [0,1] \times [0,1] \to [0,1]$ be a function defined by

$$T(\alpha, \beta) = \max(\alpha + \beta - 1, 0) \text{ for all } \alpha, \beta \in [0,1]$$

which is a t-norm. Routine calculations give that μ is a T-fuzzy subhypernear-ring of R.

PROPOSITION 3.3. *Let μ be an idempotent T-fuzzy subhypernear-ring of a hypernear-ring R. Then $\mu(x) \leq \mu(0)$ for all $x \in R$.*

Proof. For any $x \in R$, we have

$$\mu(0) \geq \inf_{\alpha \in x - x} \mu(\alpha) \geq T(\mu(x), \mu(-x)) \geq T(\mu(x), \mu(x)) = \mu(x).$$

□

PROPOSITION 3.4. *Let T be an t-norm. If μ is an idempotent T-fuzzy subhypernear-ring of hyper near-ring R, then the set*

$$R^{\omega} = \{x \in R \mid \mu(x) \geq \mu(\omega)\}$$

is a subhypernear-ring of a hyper near-ring R.

Proof. Let $x, y \in R^{\omega}$. Then $\mu(x) \geq \mu(\omega)$ and $\mu(y) \geq \mu(\omega)$. Since μ is an T-fuzzy subhypernear-ring of R, it follows that

$$\inf_{\alpha \in x+y} \{\mu(\alpha)\} \geq T(\mu(x), \mu(y)) \geq T(\mu(x), \mu(\omega)) \geq T(\mu(\omega), \mu(\omega)) = \mu(\omega).$$

Hence $x + y \subseteq R^{\omega}$ implies $x + y \in \mathcal{P}^*(R^{\omega})$. Let $x \in R^{\omega}$. Then we have $\mu(x) \geq \mu(\omega)$, and so $\mu(-x) \geq \mu(x) \geq \mu(\omega)$. Thus we have $-x \in \bar{R}^{\omega}$. Let $x, y \in R^{\omega}$. Then we get $\mu(xy) \geq T(\mu(x), \mu(y)) \geq T(\mu(\omega), \mu(\omega)) = \mu(\omega)$, and so $xy \in R^{\omega}$. This completes the proof. □

Corollary 3.5. *Let T be an t-norm. If μ is an idempotent T-fuzzy subhypernear-ring of R, then the set*

$$R^{\mu} = \{x \in R \mid \mu(x) = \mu(0)\}$$

is a subhypernear-ring of a hyper near-ring R.

Proof. From the Corollary 3.3, $R^{\mu} = \{x \in R \mid \mu(x) = \mu(0)\} = \{x \in R \mid \mu(x) \geq \mu(0)\}$, hence R^{μ} is a subhypernear-ring of a hyper near-ring R from the Prosition 3.4. □

Lemma 3.6. ([1]) *Let T be a t-norm. Then*

$$T(T(\alpha, \beta), T(\gamma, \delta)) = T(T(\alpha, \gamma), T(\beta, \delta))$$

for all $\alpha, \beta, \gamma, \delta \in [0, 1]$.

Proposition 3.7. *If μ and ν are T-fuzzy subhypernear-rings of a hypernear-ring R, then $\mu \wedge \nu : R \to [0, 1]$ defined by*

$$(\mu \wedge \nu)(x) = T(\mu(x), \nu(x))$$

for all $x \in R$ is a T-fuzzy subhypernear-ring of R.

Proof. Let $x, y \in R$. Then we have

$$\begin{aligned} \inf_{\alpha \in x+y}\{(\mu \wedge \nu)(\alpha)\} &= \inf_{\alpha \in x+y}\{T(\mu(\alpha), \nu(\alpha))\} \\ &\geq T(\inf_{\alpha \in x+y}\{\mu(\alpha)\}, \inf_{\alpha \in x+y}\{\nu(\alpha)\}) \\ &\geq T(T(\mu(x), \mu(y)), T(\nu(x), \nu(y))) \\ &= T(T(\mu(x), \nu(x)), T(\mu(y), \nu(y))) \\ &= T((\mu \wedge \nu)(x), (\mu \wedge \nu)(y)) \end{aligned}$$

and

$$\begin{aligned} (\mu \wedge \nu)(-x) &= T(\mu(-x), \nu(-x)) \geq T(\mu(x), \nu(x)) \\ &= (\mu \wedge \nu)(x) \end{aligned}$$

since $\mu(-x) \geq \mu(x)$ and $\nu(-x) \geq \nu(x)$. Also, for $x, y \in R$, we have

$$\begin{aligned} (\mu \wedge \nu)(xy) &= T(\mu(xy), \nu(xy)) \\ &= T(T(\mu(x), \mu(y)), T(\nu(x), \nu(y))) \\ &\geq T(T(\mu(x), \nu(x)), T(\mu(y), \nu(y))) \\ &= T((\mu \wedge \nu)(x), (\mu \wedge \nu)(y)) \end{aligned}$$

This completes the proof. □

Proposition 3.8. *Let H be a non-empty subset of a hypernear-ring R and let μ be a fuzzy set in R defined by*

$$\mu(x) := \begin{cases} t_1 & \text{if } x \in H \\ t_2 & \text{otherwise,} \end{cases}$$

where $t_1 > t_2$ in $[0,1]$. Then μ is an idempotent T-fuzzy subhypernear-ring of R if and only if H is a subhypernear-ring of R.

Proof. Suppose that μ is an idempotent T-fuzzy subhypernear-ring of R. Let $x, y \in H$. Then $\inf_{\alpha \in x+y} \mu(\alpha) \geq T(\mu(x), \mu(y)) = t_1$ and so $\inf_{\alpha \in x+y} \mu(\alpha) \geq t_1$. It follows that $x + y \subseteq H$. Next, let $x \in H$. Then we have $t_1 = \mu(x) \leq \mu(-x)$, and so $\mu(-x) = t_1$, that is, $-x \in H$. Next, we have $\mu(xy) \geq T(\mu(x), \mu(y)) \geq t_1$, and so $\mu(xy) = t_1$. Hence $xy \in H$ and therefore H is a subhypernear-ring of R. Conversely suppose that H is a subhypernear-ring of R. Let $x, y \in R$. If $x \in R \setminus H$ or $y \in R \setminus H$, then $\mu(x) = t_2$ or $\mu(y) = t_2$ and so

$$\inf_{\alpha \in x+y} \mu(\alpha) \geq t_2 = \min\{\mu(x), \mu(y)\} \geq T(\mu(x), \mu(y))$$

and $\mu(xy) \geq t_2 = \min\{\mu(x), \mu(y)\} \geq T(\mu(x), \mu(y))$. Assume that $x \in H$ and $y \in H$. Then $x + y \subseteq H$ and hence

$$\inf_{\alpha \in x+y} \mu(\alpha) = t_1 = \min\{\mu(x), \mu(y)\} \geq T(\mu(x), \mu(y))$$

and $\mu(xy) = t_1 = \min\{\mu(x), \mu(y)\} \geq T(\mu(x), \mu(y))$. Since $x \in H$, we obtain $-x \in H$, which implies $\mu(x) \leq \mu(-x)$. Consequently μ is a T-fuzzy subhypernear-ring of R. □

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