JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **23**, No. 2, June 2010

RESTRICTION ESTIMATES FOR ARBITRARY CONVEX CURVES IN $\mathbf{R}^2$

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Abstract. We study the restriction estimate of Fourier transform to arbitrary convex curves in R^2 with no regularity assumption. Assuming that the convex curve has the lower bound of curvatures, we extend the restriction results from smooth convex curves to arbitrary convex curves. Our work has been motivated by the lecture notes of Terence Tao. The bilinear approach and geometric observations play an important role.

1. Introduction

Given a submanifold $\mathbf{S}$ of $\mathbf{R}^d$ and a smooth measure σ on $\mathbf{S}$, we may ask that for which values of p and q the a *priori* estimate of the form

$$\|\hat{g}\|_{L^q(\mathbf{S})} \leq C_{p,q}\|g\|_{L^p(\mathbf{R}^d)},\ g \in \mathbf{S}(\mathbf{R}^d) \tag{1.1}$$

holds. Here $\mathbf{S}(\mathbf{R}^d)$ denotes the Schwartz class of rapidly decreasing smooth functions. The estimate mentioned in (1.1) is known as the restriction theorem. Although it is possible to study the restriction problem for the several kinds of submanifolds of $\mathbf{R}^d$, a unit sphere $\mathbf{S}^{d-1} \subset \mathbf{R}^d$ has been considered as one of the typical models for the problem. Moreover the restriction conjecture for the unit sphere has been investigated by many famous mathematicians. In a dual form the restriction conjecture for the unit sphere $\mathbf{S}^{d-1}$ of $\mathbf{R}^d$, $d \geq 2$, says that it has the estimate of the form

$$\|\widehat{fd\sigma}\|_{L^q(\mathbf{R}^d)} \leq C_{p,q}\|f\|_{L^p(\mathbf{S}^{d-1})} \tag{1.2}$$

whenever $q > 2d/(d-1)$ and $q \geq (d+1)p'/(d-1)$, where $p' = p/(p-1)$ is the exponent conjugate to p and $\widehat{fd\sigma}$ is the Fourier transform of the

Received December 04, 2009; Accepted April 23, 2010.
2010 Mathematics Subject Classification: Primary 42B10, Secondary 52A10.
Key words and phrases: restriction estimate, curvature, convex curve.

measure $fd\sigma$ given by

$$\widehat{fd\sigma}(\xi) = \int_{S^{d-1}} e^{2\pi i\xi x} f(x) d\sigma(x).$$

In a case where dimensions are higher than 3, the conjecture has not yet been solved. However the restriction conjecture for the circle was completely settled down due to C. Fefferman [5] and A. Zygmund [12]. More generally the restriction conjecture in two dimensions is still true for a compact subset of the smooth curve with non-vanishing curvature(see [7]). On the other hand it is not so difficult to show that it is impossible to restrict the Fourier transform to some curves containing a line segment. Therefore we may normally has a question : is it possible to get the restriction theorems for either a smooth curve or a curve containing a line segment? The purpose of this paper is to set up a new geometric method to extend the restriction results from smooth convex curves to arbitrary convex curves. In order to state our main result we need to introduce some concepts and notation. Let K be a convex curve on $\mathbf{R}^2$. Since there is a tangent at every point of K, there exists the outward normal. We denote this by $N : K \to S^1$ which is a Gauss map sending each point on the convex curve K to the outward normal. We may assume that the convex curve K doesn't contain a line segment because otherwise it is impossible to restrict the Fourier transform to the convex curve K. We shall define the curvature at each point of K. Intuitively the curvature of a plane curve at a point P can be thought as the curvature of a circle which approximates the curve most closely near that point. The curvature of a circle is directly defined by the length of its radius. The shorter the radius, the greater the curvature of the arc in the vicinity of any point P on it. The longer the radius, the bigger the circle, and less the curvature of the arc in the vicinity of any point P on it. For a very large circle the curvature of an arc at some point approaches that of a straight line i.e. zero curvature. In the following we will give the technical definition of curvature. We will find that this definition leads directly to the result that the curvature, K, of a circle is equal to the reciprocal of its radius r i.e. $K = \frac{1}{r.}$ Thus, for a circle, the length of its radius is a direct measure of its curvature.

Technical definition of curvature. Consider any smooth curve. Curvature measures the rate at which the tangent line turns per unit distance moved along the curve. Or, more simply, it measures the rate of change of direction of the curve. Let P_1 and P_2 be two points on a curve, separated by an arc of length Δs. See Fig 1. Then the average curvature of the arc from P_1 to P_2 is expressed by the fraction $\frac{\Delta\theta}{\Delta s}$ where where

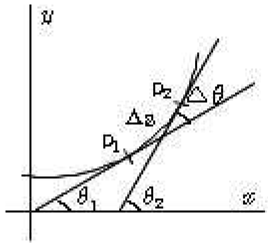

FIGURE 1. Technical definition of curvature

$\Delta\theta = \theta_2 - \theta_1$ is the angle turned through by the tangent line moving from P_1 to P_2. The curvature κ at point P is defined as

$$\kappa = \lim_{\Delta s \to 0} \frac{\Delta\theta}{\Delta s} = \frac{d\theta}{ds}. \tag{1.3}$$

We say that the curvatures of a convex curve K is bounded from below if there exist a positive constant $c > 0$ such that

$$\inf_{x \in K} \kappa(x) \geq c > 0. \tag{1.4}$$

We are ready to state our main result of restriction estimates for arbitrary convex curves with non-constant curvature in $\mathbf{R}^2$.

THEOREM 1.1. *Let $K \subset \mathbf{R}^2$ be a bounded convex curve with a non-constant and lower bounded curvature $c(> 0)$ and σ a Lebesque measure on K. Then we have*

$$\|\widehat{f d\sigma}\|_{L^q(\mathbf{R}^2)} \leq C_{p,q} \|f\|_{L^p(K, d\sigma)} \tag{1.5}$$

whenever $q > 4$ and $q > 3p'$.

Note that any regularity assumption is not required in Theorem 1.1 and the result is exactly the same to that of restriction conjecture for the circle in (1.2) except endpoints.

2. Preliminary reduction and geometric facts

Before proving our main Theorem 1.1, we will introduce the preliminary reduction and some geometric facts. As before consider a convex curve K in $\mathbf{R}^2$. We identify the unit circle S^1 with the interval $[0, 2\pi)$ and write the convex curve as follows.

$$K = \bigcup_{j=1}^{m} (N)^{-1}([2\pi(j-1)/m, 2\pi j/m)) = \bigcup_{j=1}^{m} K_j$$

where N is a Gauss map sending each point of K to the outward normal and m is a positive integer. Choose the large positive integer $m \gg 8$ so that we may assume that for each $j = 1, 2, \cdots, m$

$$\frac{\pi}{4} \gg \sup_{x,y \in K_j} |N(y) - N(x)|.$$

By the triangle inequality, it therefore suffices to consider the restriction problem for a continuous, bounded convex curve K whose the difference of any two outward normals is contained into the interval $(0, \frac{\pi}{4})$ which shall be always assumed throughout this paper. In order to decompose $K \times K$, we shall use a Whitney decomposition. For each $n \geq 0$, bisect repeatedly the segments of the convex curve K and get the 2^n segments of K with equal length $\sim 2^{-n}$. Let P_n be the set of all segment at stage $n \geq 0$. For each two segments $I, J \in P_n$ from the same stage $n \geq 2$, $I \sim J$ means that they are not adjacent, but their parents are adjacent. Note that for each $I \in P_n$, there are at most three segments $J \in P_n$ with $I \sim J$. For each $x \neq y$ on K there is exactly one pair of segments I, J containing x and y respectively such that $I \sim J$ and we can therefore write

$$(K \times K) \setminus D = \bigcup_{I \sim J} I \times J = \bigcup_{n=2}^{\infty} \Big(\bigcup_{I,J \in P_n : I \sim J} I \times J \Big) \tag{2.1}$$

where $D = \{(x, x) \in K \times K : x \in K\}$.

We shall observe some properties of the convex curve K with non-constant curvature.

Lemma 2.1. *For each $(x', y') \in K \times K$, let $z' = x' + y' \in \mathbf{R}^2$. Then there doesn't exist an element (x, y) of $K \times K$ such that $x + y = z'$ and $(x, y) \neq (x', y')$.*

Proof. Recall that we have assumed that the difference of any two outward normals on the convex curve K is contained in the interval $(0, \frac{\pi}{4})$ and the convex curve K doesn't contain any line segment. For each $z \in \mathbf{R}^2$, let

$$M_z = \{(x, y) \in \mathbf{R}^2 \times \mathbf{R}^2 : x + y = z\}.$$

By the property of a parallelogram, note that

$$(x, y) \in M_z \text{ if and only if } \frac{x + y}{2} = \frac{z}{2}. \tag{2.2}$$

Fix a $(x', y') \in K \times K$ and let $z' = x' + y' \in \mathbf{R}^2$. without loss of generality, we may assume that

$$0 < |N(y') - N(x')| < \frac{\pi}{4},$$

using the convexity of K not containing any line segment. By contradiction, we shall prove Lemma 2.1. To do this, assume that there exists a $(x, y) \in K \times K$ such that $x + y = z'$, $(x, y) \neq (x', y')$ and $0 < |N(y') - N(x')| < \frac{\pi}{4}$. Let S be an arc from x' to y' on convex curve K. If $x, y \in S$ or $x, y \notin S$, then by (2.2) and convexity, it is impossible. Thus $x \in S$ or $y \in S$. We assume that $x \in S$ and $y \notin S$. Then the straight lines of x to x' and y to y' are parallel or collinear. This is a contradiction to the fact that we have assumed that the difference of any two outward normals is contained in the interval $(0, \frac{\pi}{4})$ and $(x, y) \neq (x', y')$. Thus the proof of Lemma 2.1 is complete. □

Remark 2.2. *Let $I, J, I', J' \in P_n$. Then we easily note that Lemma 2.1 implies that if I or J is not adjacent to both I' and J' then there exists a small $\varepsilon > 0$ so that*

$$(I + J)_\varepsilon \cap (I' + J')_\varepsilon = \emptyset.$$

Therefore for every pair $(I, J) \in K \times K$ with $I \sim J$ from the same stage $n \geq 2$, there are at most eight pairs $(I', J') \in K \times K$ with $I' \sim J'$ such that

$$(I + J)_\varepsilon \cap (I' + J')_\varepsilon \neq \emptyset$$

for all $\varepsilon > 0$.

Lemma 2.3. *Let $S(R)$ be a quarter-circle with radius $R > 0$. Let H and T be two arcs in $S(R)$ of angle $\sim \theta$ and separation $\sim \theta$. Then we have that for each small $\varepsilon > 0$,*

$$\sup_{x \in \mathbf{R}^2} |H_\varepsilon \cap (T + x)| \lesssim \frac{\varepsilon}{\theta} \sim \frac{\varepsilon R}{|T|}$$

where $|T|$ means an arc length of T.

Proof. The proof is based on a geometric observation. Observe that the angle between the translated T and H_ε is $\sim \theta$. Thus we see that for all $x \in \mathbf{R}^2$,

$$|H_\varepsilon \cap (T + x)| \lesssim \frac{\varepsilon}{\sin \theta} \sim \frac{\varepsilon}{\theta}$$

which proves Lemma 2.3. □

Lemma 2.3 means that the angle between H and T doesn't largely change as it moves from H to T. Thus by (1.3) and lemma 2.3, we can obtain the followings. Let $S(R_1)$ be a quarter circle with radius $R_1(> R)$. Let $\tilde{H}$ and $\tilde{T}$ be two arcs with the same length of H and T in $S(R_1)$ of angle $\sim \theta$ and separation $\sim \theta$. Then we have that

$$\sup_{x\in\mathbf{R}^2} |H_\varepsilon \cap (T + x)| \lesssim \sup_{x\in\mathbf{R}^2} |\tilde{H}_\varepsilon \cap (\tilde{T} + x)|.$$

We can also apply the result of lemma 2.3 to any convex curve which doesn't contain any line segment.

Corollary 2.4. *Let K be a convex curve whose curvature is nonconstant and bounded from below. Then for each $I, J \in P_n$ with $I \sim J$, we have that*

$$\sup_{x\in\mathbf{R}^2} |I_\varepsilon \cap (J + x)| \lesssim \varepsilon 2^n.$$

Proof. Suppose that the convex curve K has a lower bound $c(> 0)$ of its curvature. By (1.4) we assume that the curve K has a inf $\kappa(x)$ at some point x in K. Let $\Delta\tilde{S}$ be an arc of the neighborhood of x. There exist two arcs $\tilde{I}, \tilde{J} \in P_n$ with $\tilde{I} \sim \tilde{J}$ on $\Delta\tilde{S}$ for sufficiently large n. Let I and J be two distinct arcs on any other ΔS in K. Note that the length of I and J is the same to that of $\tilde{I}$ and $\tilde{J}$. Then by lemma 2.3,

$$\sup_{x\in\mathbf{R}^2} |I_\varepsilon \cap (J + x)| \lesssim \sup_{x\in\mathbf{R}^2} |\tilde{I}_\varepsilon \cap (\tilde{J} + x)| \lesssim \frac{\varepsilon}{|\tilde{J}|} \sim \varepsilon 2^n.$$

We complete the proof. □

Theorem 2.5. *Suppose that K is a convex curve with nonconstant curvature and curvature bounded from below. Let $E \subset K$ and $I, J \in P_n$. Then*

$$\|\chi_E d_{\sigma_I} * \chi_E d_{\sigma_J}\|_{L^2(\mathbf{R}^2)} \lesssim 2^{n/2}|E \cap I|^{1/2}|E \cap J|^{1/2}$$

where d_{σ_I}, d_{σ_J} denote the restriction measure of σ to the set I, J respectively.

Proof. Using bilinear interpolation, it suffices to show that the estimates

$$\|\chi_E d_{\sigma_I} * \chi_E d_{\sigma_J}\|_{L^1(\mathbf{R}^2)} \lesssim \|\chi_E\|_{L^1(d_{\sigma_I})}\|\chi_E\|_{L^1(d_{\sigma_J})}$$

and

$$\|\chi_E d_{\sigma_I} * \chi_E d_{\sigma_J}\|_{L^\infty(\mathbf{R}^2)} \lesssim 2^n\|\chi_E\|_{L^\infty(d_{\sigma_I})}\|\chi_E\|_{L^\infty(d_{\sigma_J})}$$

hold. The first estimate is obvious by Young's inequality. To justify the second estimate, it suffices to show that

$$\|d_{\sigma_I} * d_{\sigma_J}\|_{L^\infty(\mathbf{R}^2)} \lesssim 2^n.$$

By the definition of induced Lebesque measure, it is enough to verify

$$\|\frac{1}{2\varepsilon}\chi_{I_\varepsilon} * d_{\sigma_J}\|_{L^\infty(\mathbf{R}^2)} \lesssim 2^n$$

for all sufficiently small $\varepsilon > 0$. However this is an immediate result from Corollary 2.4. Thus we complete the proof. □

3. Proof of Theorem 1.1

In order to prove the estimate (1.4) in Theorem 1.1, we may assume that f is the characteristic function of an arbitrary subset E of the convex curve K (see, [9]). By interpolation with the trivial case $q = \infty$, it therefore suffices to show that the estimate

$$\|\widehat{\chi_E d_\sigma}\|_{L^q(\mathbf{R}^2)} \lesssim \|\chi_E\|_{L^p(K,d_\sigma)} = |E|^{\frac{1}{p}} \tag{3.1}$$

holds whenever $q > 4$ and $1 - \frac{2}{q} = \frac{2}{p}$. Squaring the estimate (3.1), we shall actually show that

$$\|\widehat{\chi_E d_\sigma}\ \widehat{\chi_E d_\sigma}\|_{\frac{q}{2}} \lesssim |E|^{\frac{2}{p}}.$$

Let's estimate the left hand side. Using the fact and notation in (2.1) with triangle inequality, we see that

$$\begin{aligned}
\|\widehat{\chi_E d_\sigma}\ \widehat{\chi_E d_\sigma}\|_{\frac{q}{2}} &= \|\sum_{I\sim J}\widehat{\chi_E d_{\sigma_I}}\ \widehat{\chi_E d_{\sigma_J}}\|_{\frac{q}{2}} \\
&= \|\sum_{n\geq 2}\sum_{I,J\in P_n:I\sim J}\widehat{\chi_E d_{\sigma_I}}\ \widehat{\chi_E d_{\sigma_J}}\|_{\frac{q}{2}} \\
&\leq \sum_{n\geq 2}\|\sum_{I,J\in P_n:I\sim J}\widehat{\chi_E d_{\sigma_I}}\ \widehat{\chi_E d_{\sigma_J}}\|_{\frac{q}{2}}
\end{aligned} \tag{3.2}$$

where d_{σ_I}, d_{σ_J} denote the restriction measure of d_σ to the sets I, J respectively. Since $q > 4$, we shall estimate the $q/2$ norm by the L^∞ norm and L^2 norm using Hölder's inequality. From the triangle inequality, we see that

$$\|\sum_{I,J\in P_n:I\sim J}\widehat{\chi_E d_{\sigma_I}}\ \widehat{\chi_E d_{\sigma_J}}\|_\infty \leq \sum_{I,J\in P_n:I\sim J}|E\cap I||E\cap J|. \tag{3.3}$$

We shall estimate the right hand side by two different values. We first have the trivial estimate

$$\sum_{I,J\in P_n:I\sim J}|E\cap I||E\cap J|\leq\Big(\sum_{I\in P_n}|E\cap I|\Big)\Big(\sum_{J\in P_n}|E\cap J|\Big)=|E|^2. \tag{3.4}$$

Recall that the length of $I\in P_n$ is $\sim 2^{-n}$, and so we may estimate $|E\cap J|$ by $\sim 2^{-n}$. Since there are at most three J's for each $I\in P_n$, we have an alternative estimate

$$\begin{aligned}&\sum_{I,J\in P_n:I\sim J}|E\cap I||E\cap J|\\&\quad=\sum_{I\in P_n}|E\cap I|\Big(\sum_{I,J\in P_n:I\sim J}|E\cap J|\Big)\leq 3\cdot 2^{-n}|E|.\end{aligned} \tag{3.5}$$

Thus we obtain from (3.4) and (3.5) that

$$\sum_{I,J\in P_n:I\sim J}|E\cap I||E\cap J|\lesssim|E|\min(|E|,2^{-n}). \tag{3.6}$$

Combining this estimate with (3.3), we have

$$\|\sum_{I,J\in P_n:I\sim J}\widehat{\chi_E d_{\sigma_I}}\ \widehat{\chi_E d_{\sigma_J}}\|_\infty\lesssim|E|\min(|E|,2^{-n}). \tag{3.7}$$

In order to estimate the L^2 norm, we write

$$\|\sum_{I,J\in P_n:I\sim J}\widehat{\chi_E d_{\sigma_I}}\ \widehat{\chi_E d_{\sigma_J}}\|_{L^2}=\|\sum_{I,J\in P_n:I\sim J}\chi_E d_{\sigma_I}*\chi_E d_{\sigma_J}\|_{L^2}$$

by Plancherel's theorem. We shall show that $I\sim J$ vary then the measures $\chi_E d_{\sigma_I}*\chi_E d_{\sigma_J}$ have almost disjoint supports so that these measures are almost orthogonal. To see this, note that the support of the measure $\chi_E d_{\sigma_I}*\chi_E d_{\sigma_J}$ is contained in the set $(I+J)_\varepsilon$ for small $\varepsilon>0$, and then use the fact in Remark 2.2. Because this is almost orthogonal, we see that

$$\|\sum_{I,J\in P_n:I\sim J}\widehat{\chi_E d_{\sigma_I}}\ \widehat{\chi_E d_{\sigma_J}}\|_{L^2}\lesssim\Big(\sum_{I,J\in P_n:I\sim J}\|\chi_E d_{\sigma_I}*\chi_E d_{\sigma_J}\|_2^2\Big)^{1/2}.$$

Using theorem 2.5 and (3.6), we obtain that

$$\|\sum_{I,J\in P_n:I\sim J}\widehat{\chi_E d_{\sigma_I}}\ \widehat{\chi_E d_{\sigma_J}}\|_{L^2}\lesssim 2^{n/2}\Big(\sum_{I,J\in P_n:I\sim J}|E\cap I||E\cap J|\Big)^{1/2}$$
$$\lesssim 2^{n/2}\Big(|E|\min(|E|,2^{-n})\Big)^{1/2}.$$

Combining this with the estimate (3.7) by Hölder's inequality, we see that for $q>4$

$$\|\sum_{I,J\in P_n:I\sim J}\widehat{\chi_E d_{\sigma_I}}\ \widehat{\chi_E d_{\sigma_J}}\|_{\frac{q}{2}}\lesssim 2^{2n/q}\Big(|E|\min(|E|,2^{-n})\Big)^{1-q/2}.$$

Combining this with the estimate (3.2), simple computation yields

$$\begin{aligned}\|\widehat{\chi_E d_{\sigma_I}}\ \widehat{\chi_E d_{\sigma_J}}\|_{\frac{q}{2}} &\lesssim \sum_{n\geq 2}2^{2n/q}\Big(|E|\min(|E|,2^{-n})\Big)^{1-q/2}\\ &\leq \sum_{n\geq 2:|E|\leq 2^{-n}} 2^{2n/q}(2^{-n}|E|)^{1-q/2}+\\ &\quad \sum_{n\geq 2:|E|>2^{-n}} 2^{2n/q}(2^{-n}|E|)^{1-q/2}\\ &= |E|^{1-q/2}\sum_{n\geq 2}(2^{4/q-1})^n\lesssim |E|^{1-q/2}=|E|^{p/2}\end{aligned}$$

since $q>4$ and $1-2/q=2/p$. Thus the proof is complete.

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