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CONVERGENCE THEOREMS FOR THE C-INTEGRAL

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Abstract. In this paper, we prove convergence theorems for the C-integral.

1. Introduction and preliminaries

It is well-known [8] that the Monotone Convergence Theorem and the Dominated Convergence Theorem are valid for the Lebesgue, Perron, Denjoy and Henstock integrals. In this paper, we prove convergence theorems for the C-integral.

Throughout this paper, $I_0 = [a, b]$ is a compact interval in R. Let $D = \{(I_i, \xi_i)\}_{i=1}^n$ be a finite collection of non-overlapping tagged intervals of I_0 and let δ be a positive function on I_0.

The collection D is a δ-fine McShane partition of I_0 if $\cup_{i=1}^n I_i = I_0$, $I_i \subset (\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i))$ and $\xi_i \in I_o$ for all $i = 1, 2, ..., n$ and D is a δ-fine C_ϵ-partition of I_0 if it is a δ-fine McShane partition of I_0 and

$$\sum_{i=1}^n dist(\xi_i, I_i) < \frac{1}{\epsilon},$$

where $dist(\xi_i, I_i) = inf\{|t - \xi_i| : t \in \xi_i\}$.

Given a δ-fine partition $D = \{(I_i, \xi_i)\}_{i=1}^n$ we write

$$S(f, D) = \sum_{i=1}^n f(\xi_i)|I_i|$$

whenever $f : I_0 \to R$.

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2. Properties of the C-integral

We present the definition of the C-integral.

Definition 2.1. *A function $f : I_0 \to R$ is C-integrable if there exists a real number A such that for each $\epsilon > 0$ there is a positive function $\delta(\xi) : I_0 \to R^+$ such that*

$$|S(f, D) - A| < \epsilon$$

for each δ-fine C_ϵ-partition $D = \{(I_i, \xi_i)\}$ of I_0. The real number A is called the C-integral of f on I_0 and we write $A = \int_{I_0} f$ or $A = (C) \int_{I_0} f$.

The function f is C-integrable on the set $E \subset I_0$ if the function $f\chi_E$ is C-integrable on I_0. We write $\int_E f = \int_{I_0} f\chi_E$.

We can easily get the following theorems.

Theorem 2.2. *A function $f : I_0 \to R$ is C-integrable if and only if for each $\epsilon > 0$ there is a positive function $\delta(\xi) : I_0 \to R^+$ such that*

$$|S(f, D_1) - S(f, D_2)| < \epsilon$$

for any δ-fine C_ϵ -partitions D_1 and D_2 of I_0.

Theorem 2.3. *Let $f : I_0 \to R$.*

(1) If f is C-integrable on I_0, then f is C-integrable on every subinterval of I_0.

(2) If f is C-integrable on each of the intervals I_1 and I_2, where I_1 and I_2 are non-overlapping and $I_1 \cup I_2 = I_0$, then f is C-integrable on I_0 and $\int_{I_1} f + \int_{I_2} f = \int_{I_0} f$.

The following theorem shows that the C-integral is linear.

Theorem 2.4. *Let f and g be C-integrable functions on I_0. Then*
(1) αf is C-integrable on I_0 and $\int_{I_0} \alpha f = \alpha \int_{I_0} f$ for each $\alpha \in R$,
(2) $f + g$ is C-integrable on I_0 and $\int_{I_0}(f + g) = \int_{I_0} f + \int_{I_0} g$.

Definition 2.5. *Let $F : I_0 \to R$ and let E be a subset of I_0.*

(a) F is said to be AC_c on E if for each $\epsilon > 0$ there is a constant $\eta > 0$ and a positive function $\delta : I_0 \to R^+$ such that $|\sum_i F(I_i)| < \epsilon$ for each δ-fine partial C_ϵ-partition $D = \{(I_i, \xi_i)\}$ of I_0 satisfying $\xi_i \in E$ and $\sum_i |I_i| < \eta$.

(b) F is said to be ACG_c on E if F is continuous on E and E can be expressed as a countable union of sets on each of which F is AC_c.

Theorem 2.6. ([12]) *If a function $f : I_0 \to R$ is C-integrable on I_0 if and only if there is an ACG_c function F on I_0 such that $F' = f$ almost everywhere on I_0.*

3. Convergence Theorems for the C-integral

We will prove the convergence theorem for the C-integral.

THEOREM 3.1. (Uniform Convergence Theorem) *Let $\{f_n\}$ be a sequence of C-integral functions defined on $[a,b]$ and suppose that $\{f_n\}$ converges to f uniformly on $[a,b]$. Then f is C-integral on $[a,b]$ and*

$$\int_a^b f = \lim_{n\to\infty} \int_a^b f_n \ .$$

Proof. Given $\epsilon > 0$, there exists N such that if $n \geq N$, then

$$|f_n(x) - f(x)| < \epsilon$$

for all $x \in [a,b]$. Consequently, if $m, n \geq N$, then

$$-2\epsilon < f_n(x) - f_m(x) < 2\epsilon \ \text{for} \ x \in [a,b]$$

Hence, $-2\epsilon(b-a) < \int_a^b f_n - \int_a^b f_m < 2\epsilon(b-a)$, where $\big|\int_a^b f_n - \int_a^b f_m\big| < 2\epsilon(b-a)$. Since $\epsilon > 0$ is arbitrary, the sequence $\{\int_a^b f_n\}$ is a Cauchy sequence. Let $\lim_{n\to\infty} \int_a^b f_n = L$. If $D = \{(I_i, \xi_i)\}_{i=1}^p$ is any C_ϵ-partition of $[a,b]$ and $n \geq N$, then

$$\begin{aligned} |S(f_n, D) - S(f, D)| &= \big|\sum_{i=1}^p [f_n(\xi_i) - f(\xi_i)]|I_i|\big| \\ &\leq \sum_{i=1}^p |f_n(\xi_i) - f(\xi_i)||I_i| \\ &\leq \sum_{i=1}^p \epsilon |I_i| = \epsilon(b-a) \ . \end{aligned}$$

Choose a fixed number $n_0 \geq N$ such that $\big|\int_a^b f_{n_0} - L\big| < \epsilon$. Let δ be a positive function on $[a,b]$ such that $\big|\int_a^b f_{n_0} - S(f_{n_0}, D)\big| < \epsilon$ whenever D is a δ-fine C_ϵ-partition of $[a,b]$. Then

$$\begin{aligned} |S(f,D) - L| &\leq |S(f,D) - S(f_{n_0}, D)| + |S(f_{n_0}, D) - \int_a^b f_{n_0}| \\ &\qquad + |\int_a^b f_{n_0} - L| \\ &< \epsilon(b-a) + \epsilon + \epsilon = \epsilon(b-a+2). \end{aligned}$$

Hence, f is C-integrable on $[a,b]$ and

$$\int_a^b f = L = \lim_{n\to\infty} \int_a^b f_n.$$

□

THEOREM 3.2. (Monotone Convergence Theorem) *Let* $\{f_n\}$ *be a monotone increasing sequence of C-integrable functions defined on* $[a,b]$ *and suppose that* $\{f_n\}$ *converges pointwise to a measurable function* f *on* $[a,b]$*. If* $\lim_{n\to\infty} \int_a^b f_n$ *is finite, then* f *is* C*-integrable on* $[a,b]$ *and*

$$\int_a^b f = \lim_{n\to\infty} \int_a^b f_n.$$

Proof. Since the sequence $\{f_n\}$ is increasing, $\{f_n - f_1\}$ is an increasing sequence of nonnegative C-integrable functions on $[a,b]$. Since $f_n - f_1$ is nonnegative for each n, it follows that each $f_n - f_1$ is Lebesgue integrable on $[a,b]$ and $\lim_{n\to\infty}(f_n - f_1) = f - f_1$.

By the Monotone Convergence Theorem for the Lebesgue integral, the function $f - f_1$ is Lebesgue integrable on $[a,b]$ and

$$\begin{aligned}
(L)\int_a^b (f - f_1) &= \lim_{n\to\infty} (L)\int_a^b (f_n - f_1) \\
&= \lim_{n\to\infty} \int_a^b (f_n - f_1) \\
&= \lim_{n\to\infty} (\int_a^b f_n - \int_a^b f_1) \\
&= \lim_{n\to\infty} \int_a^b f_n - \int_a^b f_1.
\end{aligned}$$

Since $f - f_1$ and f_1 are C-integrable on $[a,b]$, the function $f = (f - f_1) + f_1$ is C-integrable on $[a,b]$. Hence,

$$\begin{aligned}
\int_a^b f &= \int_a^b (f - f_1) + \int_a^b f_1 \\
&= (L)\int_a^b (f - f_1) + \int_a^b f_1 \\
&= \lim_{n\to\infty} \int_a^b f_n - \int_a^b f_1 + \int_a^b f_1
\end{aligned}$$

$$= \lim_{n\to\infty} \int_a^b f_n.$$

□

THEOREM 3.3. (Dominated Convergence Theorem) *Let $\{f_n\}$ be a sequence of C-integrable functions defined on $[a,b]$ and suppose that $\{f_n\}$ converges to a measurable function f almost everywhere on $[a,b]$. If there exist C-integrable functions g and h on $[a,b]$ such that $g \le f_n \le h$ almost everywhere on $[a,b]$ for all n, then the function f is C-integrable on $[a,b]$ and*

$$\int_a^b f = \lim_{n\to\infty} \int_a^b f_n.$$

Proof. Since $0 \le f_n - g \le h - g$ and $h - g$ is a nonnegative C-integrable function on $[a,b]$, $h - g$ is Lebesgue integrable. Since $\{f_n\}$ converges pointwise to f almost everywhere on $[a,b]$, $0 \le f - g \le h - g$ almost everywhere on $[a,b]$. Hence, $f - g$ is Lebesgue integrable. Since $f_n - g$ converges pointwise to $f - g$ almost everywhere on $[a,b]$ and $f_n - g$ is Lebesgue integrable on $[a,b]$, by the Dominated Convergence Theorem for the Lebesgue integral we have

$$(L)\int_a^b (f-g) = \lim_{n\to\infty} (L)\int_a^b (f_n - g).$$

Since $f - g$ and g are C-integrable, $f = (f-g) + g$ is C-integrable and

$$\begin{aligned}
\int_a^b f &= \int_a^b (f-g) + \int_a^b g \\
&= (L)\int_a^b (f-g) + \int_a^b g \\
&= \lim_{n\to\infty} (L)\int_a^b (f_n - g) + \int_a^b g \\
&= \lim_{n\to\infty} \int_a^b (f_n - g) + \int_a^b g \\
&= \lim_{n\to\infty} \int_a^b f_n - \int_a^b g + \int_a^b g = \lim_{n\to\infty} \int_a^b f_n.
\end{aligned}$$

□

COROLLARY 3.4. *Let $\{f_n\}$ be a sequence of C-integrable functions defined on $[a, b]$ and suppose that $\{f_n\}$ converges to a measurable function f almost everywhere on $[a, b]$. If there exist a C-integrable function g and a Henstock integrable function h such that $g \le f_n \le h$ almost everywhere on $[a, b]$ for all n, then f is C-integrable on $[a, b]$ and*

$$\int_a^b f = \lim_{n\to\infty} \int_a^b f_n.$$

Proof. Since $0 \le f_n - g \le h - g$ and $h - g$ is a nonnegative Henstock integrable function on $[a, b]$, $h - g$ is Lebesgue integrable on $[a, b]$. Hence, $h = (h-g)+g$ is C-integrable on $[a, b]$. By Theorem 3.3, f is C-integrable on $[a, b]$ and

$$\int_a^b f = \lim_{n\to\infty} \int_a^b f_n.$$

□

We begin with the concept of uniform C-integrability. The idea behind this concept is that there exists a single positive function δ that works for all of the functions.

DEFINITION 3.5. *Let $\{f_n\}$ be a sequence of C-integrable functions defined on I_0. The sequence $\{f_n\}$ is uniformly C-integrable on I_0 if for each $\epsilon > 0$ there exists a positive function $\delta : I_0 \to R^+$ such that*

$$|S(f_n, D) - \int_{I_0} f_n| < \epsilon$$

for all n, whenever $D = \{(I_i, \xi_i)\}_{i=1}^n$ is a δ-fine C_ϵ-partition of I_0

THEOREM 3.6. *Assume that $\{f_n\}$ is uniformly C-integrable on I_0 such that*

$$\lim_{n\to\infty} f_n(\xi) = f(\xi).$$

Then the function $f : I_0 \to R$ is C-integrable on I_0 and we have

$$\lim_{n\to\infty} \int_{I_0} f_n = \int_{I_0} f.$$

Proof. Since $\{f_n\}$ is uniformly C-integrable on I_0, for each $\epsilon > 0$ there is a positive function $\delta : I_0 \to R^+$ such that

$$|S(f_n, D) - \int_{I_0} f_n| < \frac{\epsilon}{3}$$

for all n, whenever D is a δ-fine $C_{\frac{\epsilon}{3}}$-partition of I_0. Let D be a δ-fine $C_{\frac{\epsilon}{3}}$-partition of I_0. Since $\lim_{n\to\infty} f_n(\xi) = f(\xi)$, there exists an $N \in \mathbb{N}$ such that

$$|S(f_n, D) - S(f, D)| < \epsilon$$

for all $n > N$. Then we have

$$\begin{aligned}
&|\int_{I_0} f_n - \int_{I_0} f_m| \\
&\leq |S(f, D) - \int_{I_0} f_n| + |S(f, D) - \int_{I_0} f_m| \\
&\leq |S(f_n, D) - S(f, D)| + |S(f_n, D) - \int_{I_0} f_n| \\
&\quad + |S(f_m, D) - S(f, D)| + |S(f_m, D) - \int_{I_0} f_m| \\
&< \frac{8}{3}\epsilon
\end{aligned}$$

for all $m, n > N$. Hence $\{\int_{I_0} f_n\}$ is a Cauchy sequence. Let

$$\lim_{n\to\infty} \int_{I_0} f_n = A.$$

Then there exists an $M \in \mathbb{N}$ such that $|\int_{I_0} f_n - A| < \frac{\epsilon}{3}$ for all $n > M$. Take any δ-fine C_ϵ-partition $D = \{(I, \xi)\}$ of I_0. Since $\lim_{n\to\infty} f_n(\xi) = f(\xi)$, there exists a $k > M$ such that $|S(f_k, D) - S(f, D)| < \frac{\epsilon}{3}$.
Then we have

$$\begin{aligned}
&|S(f, D) - A| \\
&\leq |S(f, D) - S(f_k, D)| + |S(f_k, D) - \int_{I_0} f_k| + |\int_{I_0} f_k - A| \\
&< \epsilon
\end{aligned}$$

Hence f is C-integrable on I_0 and $\lim_{n\to\infty} \int_{I_0} f_n = \int_{I_0} f$. □

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