

ON THE STATISTICALLY COMPLETE FUZZY NORMED LINEAR SPACE.

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ABSTRACT. In this paper, we introduce the notion of the statistically complete fuzzy norm on a linear space. And we consider some relations between the fuzzy statistical completeness and ordinary completeness on a linear space.

1. Introduction

The notions of fuzzy vector spaces and fuzzy topological vector spaces were introduced in Katsaras and Liu [6]. These ideas were modified by Katsaras [4], and in [5] Katsaras defined the fuzzy norm on a vector space. In [7] Krishna and Sarma discussed the generation of a fuzzy vector topology from an ordinary vector topology on a vector space. Also Krishna and Sarma [8] observed the convergence of sequence of fuzzy points. Rhie et al.[12] introduced the notion of fuzzy α -Cauchy sequence of fuzzy points and fuzzy completeness.

In this paper, we firstly observe some properties relative to the statistically convergent sequences in a normed linear space. Secondly, we introduce the notion of the statistically complete fuzzy norm using the statistical convergence of sequences on a linear space. And we consider some relations between the fuzzy statistical completeness and the ordinary completeness on a linear space.

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2. Preliminaries

Throughout this paper, X is a vector space over the field K (R or C). Fuzzy subsets of X are denoted by Greek letters in general. χ_A denotes the characteristic function of the set A .

DEFINITION 2.1. [6] For two fuzzy subset μ_1 and μ_2 of X , the fuzzy subset $\mu_1 + \mu_2$ is defined by

$$(\mu_1 + \mu_2)(x) = \sup_{x_1+x_2=x} \min\{\mu_1(x_1), \mu_2(x_2)\}$$

And for a scalar t of K and a fuzzy subset μ of X , the fuzzy subset $t\mu$ is defined by

$$(t\mu)(x) = \begin{cases} \mu(x/t) & \text{if } t \neq 0 \\ 0 & \text{if } t = 0 \text{ and } x \neq 0 \\ \sup_{y \in X} \mu(y) & \text{if } t = 0 \text{ and } x = 0. \end{cases}$$

DEFINITION 2.2. [4] $\mu \in I^X$ is said to be

1. *convex* if $t\mu + (1-t)\mu \leq \mu$ for each $t \in [0, 1]$
2. *balanced* if $t\mu \leq \mu$ for each $t \in K$ with $|t| \leq 1$
3. *absorbing* if $\sup_{t>0} t\mu(x) = 1$ for all $x \in X$.

DEFINITION 2.3. [4] Let (X, τ) be a topological space and $\omega(\tau) = \{f : (X, \tau) \rightarrow [0, 1] \mid f \text{ is lower semicontinuous}\}$. Then $\omega(\tau)$ is a fuzzy topology on X . This topology is called the fuzzy topology generated by τ on X . The fuzzy usual topology on K means the fuzzy topology generated by the usual topology of K .

DEFINITION 2.4. [4] A *fuzzy linear topology* on a vector space X over K is a fuzzy topology on X such that the two mappings

$$\begin{aligned} + & : X \times X \rightarrow X, & (x, y) & \rightarrow x + y \\ \cdot & : K \times X \rightarrow X, & (t, x) & \rightarrow tx \end{aligned}$$

are continuous when K has the fuzzy usual topology. A linear space with a fuzzy linear topology is called a *fuzzy topological linear space* or a *fuzzy topological vector space*.

DEFINITION 2.5. [4] Let x be a point in a fuzzy topological space X . A family F of neighborhoods of x is called a base for the system of all neighborhoods of x if for each neighborhood μ of x and each $0 < \theta < \mu(x)$, there exists $\mu_1 \in F$ with $\mu_1 \leq \mu$ and $\mu_1(x) > \theta$.

DEFINITION 2.6. [5] A *fuzzy seminorm* on X is a fuzzy set ρ in X which is convex, balanced and absorbing. If in addition $\inf_{t>0} t\rho(x) = 0$ for every nonzero x , then ρ is called a *fuzzy norm*.

THEOREM 2.7. [5] If ρ is a fuzzy seminorm on X , then the family $B_\rho = \{\theta \wedge (t\rho) \mid 0 < \theta \leq 1, t > 0\}$ is a base of zero for a fuzzy linear topology τ_ρ . The fuzzy topology τ_ρ is called the *fuzzy topology induced by the fuzzy seminorm ρ* . And a linear space equipped with a fuzzy seminorm (resp. fuzzy norm) is called a *fuzzy seminormed* (resp. *fuzzy normed*) linear space.

DEFINITION 2.8. [7] Let ρ be a fuzzy seminorm on X . $P_\epsilon : X \rightarrow R_+$ is defined by $P_\epsilon(x) = \wedge\{t > 0 \mid t\rho(x) > \epsilon\}$ for each $0 < \epsilon < 1$.

THEOREM 2.9. [7] The seminorm P_ϵ is a seminorm for each $\epsilon \in (0, 1)$. Further P_ϵ is a norm on X for each $0 < \epsilon < 1$ if and only if ρ is a fuzzy norm on X .

THEOREM 2.10. [11] A metric space (X, d) is complete if and only if for any nested sequence $A_1 \supset A_2 \supset \dots$ of nonempty closed sets of X such that $\text{diam} A_n \rightarrow 0$, $\bigcap_{n \in \mathbb{Z}} A_n \neq \emptyset$.

3. Statistical convergence on a normed linear space.

In [2], H. Fast introduced an extension of the usual concept of sequential limits which he called statistical convergence. In [14] I. J. Schoenberg gave some basic properties of statistical convergence and also studied the concept as a summability method. In [15], one may find a recent trend for this topic. In this section, we prove that every statistical Cauchy sequence on a Banach space is statistically convergent ; as an extension of Fridy's [3] result to a normed linear space.

DEFINITION 3.1. [15] The natural density of a positive integer set K is defined by $\delta(K) = \lim_{n \rightarrow \infty} \frac{1}{n} |\{k \in K : k \leq n\}|$, where $|\{k \in K : k \leq n\}|$ is the number of elements of K not exceeding n .

It is clear that for a finite set K , we have $\delta(K) = 0$. The natural density may not exist for each set K and is different from zero which means $\delta(K) > 0$. Besides that, $\delta(K^c) = 1 - \delta(K)$ where K^c means the complement of K .

NOTATION. For facilitation, we introduce the following notation: if $\langle x_k \rangle$ is a sequence such that x_k satisfies property P for all k except a set of natural density zero (equivalently for all k in a positive integer set

with natural density one), then we say that x_k satisfies P for “almost all k ”, and we abbreviate this by “*a.a. k.*”

DEFINITION 3.2. The sequence $\langle x_k \rangle$ on a normed linear space is statistically convergent to the vector x provided that for each $\epsilon > 0$,

$$\lim_{n \rightarrow \infty} \frac{1}{n} |\{k \in n : \|x_k - x\| \geq \epsilon\}| = 0,$$

i.e.,

$$(*) \quad \|x_k - x\| < \epsilon \quad a.a. \quad k.$$

In this case we write $st - \lim x_k = x$.

Example [3] Define $x_k = x$ if k is a square and $x_k = 0$ otherwise. Then $|\{k \leq n : x_k \neq 0\}| \leq \sqrt{n}$, so $st - \lim x_k = 0$. Note that we could have assigned any values whatsoever to x_k when k is a square, and we would still have $st - \lim x_k = 0$. It is clear that if the inequality in $(*)$ holds for all but finitely many k , then $\lim x_k = x$. It follows that $\lim x_k = x$ implies $st - \lim x_k = x$. As most convergence theories, we introduce the statistical analogue of the Cauchy convergence criterion.

DEFINITION 3.3. The sequence $\langle x_k \rangle$ on a normed linear space is statistical Cauchy sequence if for every $\epsilon > 0$ there exists a number $N(= N(\epsilon))$ such that

$$\|x_k - x_N\| < \epsilon \quad a.a. \quad k,$$

i.e.,

$$\lim_{n \rightarrow \infty} \frac{1}{n} |\{k \leq n : \|x_k - x_N\| \geq \epsilon\}| = 0,$$

THEOREM 3.4. Every statistically convergent sequence is a statistical Cauchy sequence on a normed linear space.

Proof. Suppose $st - \lim x_k = x$ and $\epsilon > 0$. Then $\|x_k - x\| < \epsilon/2$ *a.a. k* and if N is chosen so that $\|x_N - x\| < \epsilon/2$ then we have

$$\begin{aligned} \|x_k - x_N\| &< \|x_k - x\| + \|x_N - x\| \\ &< \epsilon/2 + \epsilon/2 \quad a.a. \quad k. \end{aligned}$$

Hence, $\langle x_k \rangle$ is a statistically Cauchy sequence. $\square$

THEOREM 3.5. For every statistical Cauchy sequence $\langle x_k \rangle$ on a complete normed linear space, there is a convergent sequence $\langle y_k \rangle$ such that $x_k = y_k$ *a.a. k.*

Proof. Let $\langle x_k \rangle$ be a statistical Cauchy sequence. Then we can choose N so that the closed ball $C = \{x \in X : \|x_N - x\| \leq 1\}$ contains x_k a.a. k . Also we can choose M so that $C' = \{x \in X : \|x_M - x\| \leq 1/2\}$ contains x_k a.a. k . We assert that $C_1 = C \cap C'$ contains x_k a.a. k .

For,

$$\begin{aligned} & \{k \leq n : x_k \notin C \cap C'\} \\ &= \{k \leq n : x_k \notin C\} \cup \{k \leq n : x_k \notin C'\}, \end{aligned}$$

so

$$\begin{aligned} & \lim_{n \rightarrow \infty} \frac{1}{n} |\{k \leq n : x_k \notin C \cap C'\}| \\ & \leq \lim_{n \rightarrow \infty} \frac{1}{n} |\{k \leq n : x_k \notin C\}| + \lim_{n \rightarrow \infty} \frac{1}{n} |\{k \leq n : x_k \notin C'\}| = 0. \end{aligned}$$

Therefore C_1 is a closed set of diameter less than or equal to 1 that contains x_k a.a. k . Now we proceed by choosing $N(2)$ so that $C'' = \{x \in X : \|x_{N(2)} - x\| \leq 1/4\}$ contains x_k a.a. k , and by the preceding argument $C_2 = C_1 \cap C''$ contains x_k a.a. k , and C_2 has the diameter less than or equal to $1/2$. Continuing inductively we construct a sequence $\{C_m\}_{m=1}^{\infty}$ of closed sets such that for each m , $C_m \supseteq C_{m+1}$, the diameter of C_m is not greater than 2^{1-m} , and $x_k \in C_m$ a.a. k . By Theorem 2.10, there is a vector y such that $\bigcap_{m=1}^{\infty} C_m = \{y\}$. Using the fact that $x_k \in C_m$ a.a. k we choose an increasing positive integer sequence $\{T_m\}_{m=1}^{\infty}$ such that

$$(*) \quad \frac{1}{n} |\{k \leq n : x_k \notin C_m\}| < \frac{1}{m} \quad \text{if } n > T_m.$$

Now define a subsequence z of $\langle x_k \rangle$ consisting of terms x_k such that $k > T_1$ and

$$\text{if } T_m < k < T_{m+1} \quad \text{then } x_k \notin C_m.$$

Next define a sequence $\langle y_k \rangle$ by

$$y_k = \begin{cases} y & \text{if } x_k \text{ is a term of } z, \\ x_k & \text{otherwise} \end{cases}$$

Then $\lim y_k = y$; for, if $\epsilon > 1/m > 0$ and $k > T_m$ then either x_k is a term of z , which means $y_k = y$, or $y_k = x_k \in C_m$ and $\|y_k - y\| \leq \text{diameter of } C_m \leq 2^{1-m}$. We also assert that $x_k = y_k$ a.a. k . To verify this we observe that if $T_m < n < T_{m+1}$ then

$$\{k \leq n : y_k \neq x_k\} \subseteq \{k \leq n : x_k \notin C_m\},$$

so by $(*)$

$$\frac{1}{n} |\{k \leq n : y_k \neq x_k\}| \leq \frac{1}{n} |\{k \leq n : x_k \notin C_m\}| < \frac{1}{m}.$$

Hence, the limit as $n \rightarrow \infty$ is 0 and $x_k = y_k$ a.a. k . This completes the proof. $\square$

THEOREM 3.6. *If $\langle x_k \rangle$ is a sequence on a Banach space for which there is a convergent sequence $\langle y_k \rangle$ such that $x_k = y_k$ a.a. k , then it is a statistically convergent sequence.*

Proof. Let $x_k = y_k$ a.a. k and $\lim y_k = L$. Suppose $\epsilon > 0$. Then for each n ,

$$\begin{aligned} & \{k \leq n : \|x_k - L\| \geq \epsilon\} \\ & \subseteq \{k \leq n : x_k \neq y_k\} \cup \{k \leq n : \|y_k - L\| > \epsilon\}; \end{aligned}$$

since $\lim y_k = L$, the latter set contains a fixed number of integers, say $l = l(\epsilon)$. Therefore

$$\begin{aligned} & \lim_{n \rightarrow \infty} \frac{1}{n} |\{k \leq n : \|x_k - L\| \geq \epsilon\}| \\ & \leq \lim_{n \rightarrow \infty} \frac{1}{n} |\{k \leq n : x_k \neq y_k\}| + \lim_{n \rightarrow \infty} \frac{l}{n} = 0 \text{ because } x_k = y_k \text{ a.a. } k. \end{aligned}$$

Hence, $\|x_k - L\| < \epsilon$ a.a. k , so the proof is complete. $\square$

4. Fuzzy statistical convergence and fuzzy statistical completeness.

In this section, we introduce the notion of a statistically complete fuzzy norm on a linear space. And we consider some relations between the fuzzy statistical completeness and the ordinary completeness. Now, we define the statistical convergence of sequences in a fuzzy normed linear space.

DEFINITION 4.1. Let (X, ρ) be a fuzzy normed linear space. A sequence $\langle x_k \rangle \subset X$ is said to statistically converge to $x \in X$ if for every $t > 0$ and $0 < \epsilon < 1$, there exists a positive integer set K with natural density one such that $k \in K$ implies $t\rho(x_k - x) > 1 - \epsilon$, i.e., $t\rho(x_k - x) > 1 - \epsilon$ a.a. k .

THEOREM 4.2. Let (X, ρ) be a fuzzy normed linear space. A sequence $\langle x_k \rangle \subset X$ statistically converges to $x \in X$ if and only if for every $t > 0$ and $0 < \epsilon < 1$, $P_{1-\epsilon}(x_k - x) < t$ a.a. k .

Proof. Let $t > 0$ and $\epsilon \in (0, 1)$ be given. Since $\langle x_k \rangle$ statistically converges to x , there exists a positive integer set K with natural density one such that

$$\begin{aligned} x \in K \text{ implies } \frac{t}{2} \rho(x_k - x) &> 1 - \epsilon \\ \implies P_{1-\epsilon}(x_n - x) &\leq \frac{t}{2} < t \quad a.a. \quad k. \end{aligned}$$

For the converse, let $t > 0$ and $\epsilon > 0$ be given. Then

$$\begin{aligned} \implies P_{1-\epsilon}(x_n - x) &< t \quad a.a. \quad k \\ \implies t' \rho(x_n - x) &> 1 - \epsilon \quad a.a. \quad k \\ &\quad \text{for some } t' \in (P_{1-\epsilon}(x_n - x), t) \\ \implies t \rho(x_n - x) &\geq t' \rho(x_n - x) > 1 - \epsilon \quad a.a. \quad k. \end{aligned}$$

This completes the proof. □

The next definition of the statistical Cauchy sequence in a fuzzy normed linear space is an extension of the fuzzy Cauchy sequence.

DEFINITION 4.3. Let (X, ρ) be a fuzzy normed linear space. A sequence $\langle x_k \rangle \subset X$ is a statistical Cauchy sequence if and only if for every $t > 0$ and $0 < \epsilon < 1$, there exists a positive integer set K with natural density one such that $k, l \in K$ implies $t \rho(x_k - x_l) > 1 - \epsilon$, i.e., $t \rho(x_k - x_l) > 1 - \epsilon \quad a.a. \quad k, l$.

THEOREM 4.4. Let (X, ρ) be a fuzzy normed linear space. A sequence $\langle x_k \rangle \subset X$ is a statistical Cauchy sequence if and only if for every $t > 0$ and $0 < \epsilon < 1$, $P_{1-\epsilon}(x_k - x_l) < t \quad a.a. \quad k, l$.

Proof. The proof is similar to that of Theorem 4.2. We omit it. □

The following theorem is easily verified with elementary skill from Theorem 4.2. and Theorem 4.4.

THEOREM 4.5. Every statistically convergent sequence in a fuzzy normed linear space is a statistical Cauchy sequence.

Now, we introduce the statistically complete fuzzy norm using the statistical Cauchy sequence defined above.

DEFINITION 4.6. A fuzzy normed linear space (X, ρ) is said to be fuzzy statistically complete if every statistical Cauchy sequence in X statistically converges to a point in X .

LEMMA 4.7. *Let $(X, \|\cdot\|)$ be a normed linear space and B the closed unit ball of X . Then every statistical Cauchy sequence in the fuzzy normed linear space (X, χ_B) is a statistical Cauchy sequence with respect to the ordinary norm.*

Proof. Let $\langle x_k \rangle \subset X$ be a statistical Cauchy sequence on (X, χ_B) and $\delta > 0$. Since $\langle x_k \rangle$ is a statistical Cauchy sequence, for this δ and for every $0 < \epsilon < 1$, there exists a positive integer set K with natural density one such that $k, l \in K$ implies

$$\begin{aligned} & \frac{\delta}{2} \chi_B(x_k - x_l) > 1 - \epsilon \\ \implies & \chi_B\left(\frac{\delta}{2}(x_k - x_l)\right) > 1 - \epsilon \quad a.a. \quad k, l \\ \implies & \chi_B\left(\frac{\delta}{2}(x_k - x_l)\right) = 1 \quad a.a. \quad k, l \\ \implies & \|x_k - x_l\| \leq \frac{\delta}{2} < \delta \quad a.a. \quad k, l. \end{aligned}$$

Therefore $\langle x_k \rangle$ is a statistical Cauchy sequence in $(X, \|\cdot\|)$. This prove the lemma. $\square$

THEOREM 4.8. *Let $(X, \|\cdot\|)$ be a Banach space. Then the fuzzy normed linear space (X, χ_B) is fuzzy statistically complete where B is the closed unit ball of X .*

Proof. Let $\langle x_k \rangle$ be a statistically Cauchy sequence in (X, χ_B) . Then it is a statistical Cauchy sequence with respect to the ordinary norm $\|\cdot\|$ by the above lemma. Since $(X, \|\cdot\|)$ is a Banach space, there exists an $x \in X$ such that x_k statistically converges to x by Theorem 3.5 and 3.6. Now, we show that $\langle x_k \rangle$ statistically converges to this x in (X, χ_B) . Let $t > 0$ and $0 < \epsilon < 1$. Then there exists a positive integer set K with natural density one such that

$$\begin{aligned} & k \in K \text{ implies } \|x_k - x\| < t \\ \implies & \left\| \frac{1}{t}(x_k - x) \right\| < 1 \quad a.a. \quad k \\ \implies & \chi_B\left(\frac{1}{t}(x_k - x)\right) = 1 \quad a.a. \quad k \\ \implies & t \chi_B(x_k - x) > 1 - \epsilon \quad a.a. \quad k \end{aligned}$$

That is $\langle x_k \rangle$ statistically converges to x , therefore (X, χ_B) is fuzzy statistically complete. This completes the proof. $\square$

COROLLARY 4.9. *The field $K(R$ or $C)$ with the fuzzy topology generated by the usual topology on K is a fuzzy statistically complete fuzzy normed linear space.*

DEFINITION 4.10. [5] Two fuzzy seminorms ρ_1, ρ_2 on X are said to be *equivalent* iff $\tau_{\rho_1} = \tau_{\rho_2}$.

THEOREM 4.11. [12] Let $(X, \|\cdot\|)$ be a normed linear space. If ρ is a lower semi-continuous fuzzy norm on X , and has the bounded support: $\{x \in X \mid \rho(x) > 0\}$ is bounded, then ρ is equivalent to the fuzzy norm χ_B where B is the closed unit ball of X .

By Theorem 4.8. and 4.11 we get the following theorem.

THEOREM 4.12. If X is a Banach space and ρ is a lower semicontinuous fuzzy norm on X having the bounded support, then the fuzzy normed linear space (X, ρ) is fuzzy statistically complete.

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