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MEET-REDUCIBILITY OF TL-SUBGROUPS

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Abstract. The structure of a TL-subgroup can be understood from the representations of the TL-sub group as meets of TL-subgroups containing the TL-subgroup. Indeed, the structure of the meet of TL-subgroups can easily be obtained from the structures of the TL-subgroups and the structures of the TL-subgroups may be more simple than the structure of the meet. In this paper, we discuss meet-reducibility of TL-subgroups.

1. Introduction

Zadeh [7] introduced the concept of fuzzy subsets and Rosenfeld [4] introduced the concept of fuzzy subgroups. Following these ideas, many authors are engaged in generalizing various notions of group theory in the fuzzy setting. And Yu et al. [6] introduced and studied the concept of TL-subgroups that is an extension of the concept of fuzzy subgroups.

One way to grasp the structure of a group is representing the group as the direct product of its subgroups of which structures are more simple than the structure of the group. In parallel, the structure of a TL-subgroup can be understood from the representations of the TL-subgroup as meets of TL-subgroup containing the TL-subgroup. Indeed, the structure of the meet of TL-subgroups can easily be obtained from the structures of the TL-subgroups and the structures of the TL-subgroups may be more simple than the structure of the meet. In this paper, we discuss meet-reducibility of TL-subgroups.

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2. Preliminaries

Throughout this paper, we let L denote a complete lattice that contains at least two distinct elements. The meet, join, and partial ordering will be written as $\wedge$, $\vee$, and $\leq$, respectively. We also write 1 for the greatest element of L.

Definition 2.1 ([5]). A binary operation T on L is called a t-norm if it satisfies the following conditions;
(1) $(aTb)Tc = aT(bTc)$,
(2) $aTb = bTa$,
(3) if $b \leq c$, then $aTb \leq aTc$,
(4) $aT1 = a$,
where a, b, $c \in L$.

The meet $\wedge$ on L is a t-norm. From now on we will always assume that $T = \wedge$. We will write the identity element of a group G by e and the order of x in G by $O(x)$.

Definition 2.2 ([6]). An L-subset μ of a group G, i.e., a function μ from G to L, is called a TL-group or a TL-subgroup of G if it satisfies the following conditions;
(1) $\mu(e) = 1$,
(2) $\mu(x^{-1}) \geq \mu(x)$ for all $x \in G$,
(3) $\mu(xy) \geq \mu(x)T\mu(y)$ for all $x, y \in G$.

Note that the concept of a TL-subgroup is an extension of the concept of a fuzzy subgroup. We will write the set $\{x \in G | \mu(x) = 1\}$ by G_μ, where μ is a TL-subgroup of a group G.

Definition 2.3 ([3]). Let μ be a TL-subgroup of a group G. For a given $x \in G$, the least positive integer n such that $\mu(x^n) = 1$ is said to be the (TL-)order of x with respect to μ (briefly, $O_\mu(x)$). If no such n exists, x is said to have infinite (TL-)order with respect to μ.

Definition 2.4 ([2]). Let μ be a TL-subgroup of an Abelian group G. μ is said to be torsion if $O_\mu(x)$ is finite for all $x \in G$.

Definition 2.5 ([2]). Let μ be a TL-subgroup of a group G. For a prime p, μ is called a TL-p-subgroup of G if $O_\mu(x)$ is a power of p for every $x \in G$.

When $\{\mu_i | i \in I\}$ is a set of TL-subgroups of a group G, the meet $\mu = \wedge\{\mu_i | i \in I\}$ of $\{\mu_i | i \in I\}$ is the TL-subgroup of G defined by $\mu(x) = \wedge\{\mu_i(x) | i \in I\}$.

3. Meet-reducibility of TL-subgroups

A TL-subgroup μ of a group G is said to be meet-reducible if there exist TL-subgroups ν and η of G such that $\mu \neq \nu$, $\mu \neq \eta$, and $\mu = \nu \wedge \eta$. The meet-reducibility of a TL-subgroup heavily depends on the number of its values.

LEMMA 3.1. *Let μ be a TL-subgroup of a group G. Let μ take more than or equal to* 3 *values. If there exist at least two compatible values among the values except the greatest element* 1 *of L, then μ is meet-reducible.*

Proof. Let a_1 and a_2 be two compatible values of μ with $1 > a_1 > a_2$. Define L-subsets ν and η of G as follows:

$$\nu(x) = \begin{cases} 1 & \text{if } \mu(x) \geq a_1, \\ \mu(x) & \text{otherwise,} \end{cases}$$

and

$$\eta(x) = \begin{cases} \mu(x) & \text{if } \mu(x) \geq a_1, \\ a_1 & \text{otherwise.} \end{cases}$$

Then ν and η are TL-subgroups of G with $\mu \neq \nu$, $\mu \neq \eta$, and $\mu = \nu \wedge \eta$. So μ is meet-reducible. □

LEMMA 3.2. *Let μ be a TL-subgroup of a group G. Let μ take more than or equal to* 3 *values. If any two values of μ except* 1 *are not compatible, then μ is meet-reducible.*

Proof. Let a_1 and a_2 be any two different values of μ not equal to 1. Define L-subsets ν and η of G as follows:

$$\nu(x) = \begin{cases} 1 & \text{if } \mu(x) \geq a_1, \\ \mu(x) & \text{otherwise,} \end{cases}$$

and

$$\eta(x) = \begin{cases} 1 & \text{if } \mu(x) \geq a_2, \\ \mu(x) & \text{otherwise.} \end{cases}$$

Then ν and η are TL-subgroups of G with $\mu \neq \nu$, $\mu \neq \eta$, and $\mu = \nu \wedge \eta$. So μ is meet-reducible. □

By Lemma 3.1 and 3.2, we have the following theorem.

THEOREM 3.3. *Let μ be a TL-subgroup of a group G. If μ takes more than or equal to* 3 *values, then μ is meet-reducible.*

If a TL-subgroup takes exactly one value, then it is clearly meet-irreducible. Now we will concern with TL-subgroups which take exactly two values. Recall that a subgroup T of a group G is said to be meet-reducible if there exist subgroups H and K of G such that $T \neq H$, $T \neq K$, and $T = H \cap K$.

THEOREM 3.4. *Let μ be a TL-subgroup of a group G. Let μ take exactly two values. If G_μ is meet-reducible, then μ is meet-reducible.*

Proof. Let 1 and a be the two values of μ. Suppose that G_μ is meet-reducible. Then there exist subgroups H and K of G such taht $G_\mu \neq H$, $G_\mu \neq K$, and $G_\mu = H \cap K$. Define L-subsets ν and η of G as follows:

$$\nu(x) = \begin{cases} 1 & \text{if } x \in H, \\ a & \text{otherwise,} \end{cases}$$

and

$$\eta(x) = \begin{cases} 1 & \text{if } x \in K, \\ a & \text{otherwise.} \end{cases}$$

Then ν and η are TL-subgroups of G with $\mu \neq \nu$, $\mu \neq \eta$, and $\mu = \nu \wedge \eta$. So μ is meet-reducible. □

Recall that an element a of L is meet-reducible if there exist elements b and c of L such that $a \neq b$, $a \neq c$, and $a = b \wedge c$.

THEOREM 3.5. *Let μ be a TL-subgroup of a group G. Let μ take exactly two values* 1 *and a. If a is meet-reducible, then μ is meet-reducible.*

Proof. Suppose that a is meet-reducible. Then there exist elements b and c of L such that $a \neq b$, $a \neq c$, and $a = b \wedge c$. Define L-subsets ν and η of G as follows:

$$\nu(x) = \begin{cases} 1 & \text{if } x \in G_\mu, \\ b & \text{otherwise,} \end{cases}$$

and

$$\eta(x) = \begin{cases} 1 & \text{if } x \in G_\mu, \\ c & \text{otherwise.} \end{cases}$$

Then ν and η are TL-subgroups of G with $\mu \neq \nu$, $\mu \neq \eta$, and $\mu = \nu \wedge \eta$. So μ is meet-reducible. □

Let μ be a TL-subgroup of a group G. If there exists a minimal TL-p-subgroup of G containing μ, then it is unique because the meet of TL-p-subgroups of G is obviously a TL-p-subgroup. We will denote it by $\mu_{(p)}$. Note that $\mu_{(p)}$ does not exist in general even if $L = [0, 1]$ [1].

THEOREM 3.6 ([2]). *Let μ be a torsion TL-subgroup of an Abelian group G. Then* $\mu = \bigwedge_p \mu_{(p)}$

Now we have a corollary from Theorem 3.6.

COROLLARY 3.7. *Let μ be a torsion TL-subgroup of an Abelian group. If there exist two primes p and q such that $\mu_{(p)}$ and $\mu_{(q)}$ are not trivial, then μ is meet-reducible.*

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