

ON THE 2-VARIABLE SUBNORMAL COMPLETION PROBLEM

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ABSTRACT. In this note we give a connection between the truncated moment problem and the 2-variable subnormal completion problem.

1. Preliminaries

Let $\mathcal{H}$ be a complex Hilbert space and let $\mathcal{B}(\mathcal{H})$ denote the algebra of bounded linear operators on $\mathcal{H}$. Recall that a bounded linear operator $T \in \mathcal{B}(\mathcal{H})$ is *normal* if $T^*T = TT^*$, and *subnormal* if $T = N|_{\mathcal{H}}$, where N is normal and $N(\mathcal{H}) \subseteq \mathcal{H}$. An operator T is said to be *hyponormal* if $T^*T \geq TT^*$. For $S, T \in \mathcal{B}(\mathcal{H})$, let $[S, T] := ST - TS$. An n -tuple $\mathbf{T} := (T_1, \dots, T_n)$ of operators on $\mathcal{H}$ is said to be (jointly) *hyponormal* if the operator matrix

$$[\mathbf{T}^*, \mathbf{T}] := \begin{pmatrix} [T_1^*, T_1] & [T_2^*, T_1] & \cdots & [T_n^*, T_1] \\ [T_1^*, T_2] & [T_2^*, T_2] & \cdots & [T_n^*, T_2] \\ \vdots & \vdots & \ddots & \vdots \\ [T_1^*, T_n] & [T_2^*, T_n] & \cdots & [T_n^*, T_n] \end{pmatrix}$$

is positive semidefinite on the direct sum of n copies of $\mathcal{H}$ (cf. [1], [9]). For instance, if $n = 2$,

$$[\mathbf{T}^*, \mathbf{T}] := \begin{pmatrix} [T_1^*, T_1] & [T_2^*, T_1] \\ [T_1^*, T_2] & [T_2^*, T_2] \end{pmatrix}.$$

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The n -tuple $\mathbf{T} \equiv (T_1, T_2, \dots, T_n)$ is said to be *normal* if $\mathbf{T}$ is commuting and each T_i is normal, and $\mathbf{T}$ is *subnormal* if $\mathbf{T}$ is the restriction of a normal n -tuple to a common invariant subspace. In particular, a commuting pair $\mathbf{T} \equiv (T_1, T_2)$ is said to be *k-hyponormal* ($k \geq 1$) ([10]) if

$$\mathbf{T}(k) := (T_1, T_2, T_1^2, T_2 T_1, T_2^2, \dots, T_1^k, T_2 T_1^{k-1}, \dots, T_2^k)$$

is hyponormal, or equivalently

$$[\mathbf{T}(k)^*, \mathbf{T}(k)] = [(T_2^q T_1^p)^*, T_2^m T_1^n]_{\substack{1 \leq n+m \leq k \\ 1 \leq p+q \leq k}} \geq 0.$$

Clearly, normal $\Rightarrow$ subnormal $\Rightarrow$ k -hyponormal. We now review results for one variable subnormal completion. For $\alpha \equiv \{\alpha_n\}_{n=0}^\infty$ a bounded sequence of positive real numbers (called *weights*), let $W_\alpha : \ell^2(\mathbb{Z}_+) \rightarrow \ell^2(\mathbb{Z}_+)$ be the associated unilateral weighted shift, defined by $W_\alpha e_n := \alpha_n e_{n+1}$ (all $n \geq 0$), where $\{e_n\}_{n=0}^\infty$ is the canonical orthonormal basis in $\ell^2(\mathbb{Z}_+)$. For a weighted shift W_α , the *moments of α* are given as

$$\gamma_k \equiv \gamma_k(\alpha) := \begin{cases} 1, & \text{if } k = 0 \\ \alpha_0^2 \cdots \alpha_{k-1}^2, & \text{if } k > 0. \end{cases}$$

It is easy to see that W_α is never normal, and that it is hyponormal if and only if $\alpha_0 \leq \alpha_1 \leq \dots$. C. Berger's characterization of subnormality for unilateral weighted shifts (cf. [4], [2, II.6.10]) states that W_α is subnormal if and only if there exists a Borel probability measure (so called Berger measure) μ supported in $[0, \|W_\alpha\|]$, with $\|W_\alpha\| \in \text{supp } \mu$, such that

$$\gamma_n = \int t^{2n} d\mu(t) \quad \text{for all } n \geq 0.$$

In 1966, Stampfli [16] explicitly exhibited for a subnormal weighted shift A_0 its minimal normal extension

$$N := \begin{pmatrix} A_0 & B_1 & & 0 \\ & A_1 & B_2 & \\ & & A_2 & \ddots \\ 0 & & & \ddots \end{pmatrix},$$

where A_n is a weighted shift with weights $\{a_0^{(n)}, a_1^{(n)}, \dots\}$, $B_n := \text{diag}\{b_0^{(n)}, b_1^{(n)}, \dots\}$, and these entries satisfy:

- (I) $(a_j^{(n)})^2 - (a_{j-1}^{(n)})^2 + (b_j^{(n)})^2 \geq 0$ ($b_j^{(0)} = 0$ for all j);
- (II) $b_j^{(n)} = 0 \implies b_{j+1}^{(n)} = 0$;

(III) there exists a constant M such that $|a_j^{(n)}| \leq M$ and $|b_j^{(n)}| \leq M$ for $n = 0, 1, \dots$ and $j = 0, 1, \dots$.

Here $b_j^{(n+1)} := [(a_j^{(n)})^2 - (a_{j-1}^{(n)})^2 + (b_j^{(n)})^2]^{\frac{1}{2}}$ and $a_j^{(n+1)} := a_j^{(n)} \frac{b_{j+1}^{(n+1)}}{b_j^{(n+1)}}$

(if $b_{j_0}^{(n)} = 0$, then $a_{j_0}^{(n)}$ is taken to be 0).

Thus, we have W_α is subnormal if and only if conditions (I), (II), (III) hold for W_α .

Given an initial segment of weights $\alpha : \alpha_0, \dots, \alpha_m$, a sequence $\hat{\alpha} \in \ell^\infty(\mathbb{Z}_+)$ such that $\hat{\alpha}_i = \alpha_i$ ($i = 0, \dots, m$) is said to be *recursively generated* by α if there exist $r \geq 1$ and $\varphi_0, \dots, \varphi_{r-1} \in \mathbb{R}$ such that $\gamma_{n+r} = \varphi_0 \gamma_n + \dots + \varphi_{r-1} \gamma_{n+r-1}$ (all $n \geq 0$), where $\gamma_0 := 1$, $\gamma_n := \alpha_0^2 \dots \alpha_{n-1}^2$ ($n \geq 1$); in this case $W_{\hat{\alpha}}$ is said to be *recursively generated*. If the associated recursively generated weighted shift $W_{\hat{\alpha}}$ is subnormal, then its Berger measure is a finitely atomic measure of the form $\mu := \rho_0 \delta_{s_0} + \dots + \rho_{r-1} \delta_{s_{r-1}}$. Let $\alpha : \alpha_0, \dots, \alpha_m$ ($m \geq 0$) be an initial segment of positive weights and let $\omega = \{\omega_n\}_{n=0}^\infty$ be a bounded sequence of positive numbers. We say that W_ω is a *completion* of α if $\omega_n = \alpha_n$ ($0 \leq n \leq m$), and we write $\alpha \subset \omega$. The *completion problem* for a property (P) entails finding necessary and sufficient conditions on α to ensure the existence of a weight sequence $\omega \supset \alpha$ such that W_ω satisfies (P). In [2, Theorem 3.5], the following criterion was established.

THEOREM 1. (Subnormal Completion Criterion) *If $\alpha : \alpha_0, \dots, \alpha_n$ ($n \geq 0$) is an initial segment of positive weights then the following are equivalent:*

- (i) α has a subnormal completion;
- (ii) α has a recursively generated subnormal completion;
- (iii) the Hankel matrices $H(l)$ and $H_x(m-1)$ are both positive ($l := \lceil \frac{n+1}{2} \rceil$ and $m := \lfloor \frac{n}{2} \rfloor + 1$) and the vector

$$\begin{pmatrix} \gamma_{l+1} \\ \vdots \\ \gamma_{2l+1} \end{pmatrix} \quad (\text{resp.} \quad \begin{pmatrix} \gamma_{m+1} \\ \vdots \\ \gamma_{2m} \end{pmatrix})$$

is in the range of $H(l)$ (resp. $H_x(m-1)$) when n is even (resp. odd). Here

$$H(j) := \begin{pmatrix} \gamma_0 & \gamma_1 & \cdots & \gamma_j \\ \gamma_1 & \gamma_2 & \cdots & \gamma_{j+1} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_j & \gamma_{j+1} & \cdots & \gamma_{2j} \end{pmatrix}, \quad H_x(j) := \begin{pmatrix} \gamma_1 & \gamma_2 & \cdots & \gamma_{j+1} \\ \gamma_2 & \gamma_3 & \cdots & \gamma_{j+2} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{j+1} & \gamma_{j+2} & \cdots & \gamma_{2j+1} \end{pmatrix}.$$

We now consider double-indexed positive bounded sequences $\alpha_{\mathbf{k}}, \beta_{\mathbf{k}} \in \ell^\infty(\mathbb{Z}_+^2)$, $\mathbf{k} \equiv (k_1, k_2) \in \mathbb{Z}_+^2$. Define the 2-variable weighted shift $W_{(\alpha, \beta)} \equiv (T_1, T_2)$ by

$$T_1 e_{\mathbf{k}} := \alpha_{\mathbf{k}} e_{\mathbf{k} + \varepsilon_1} \quad \text{and} \quad T_2 e_{\mathbf{k}} := \beta_{\mathbf{k}} e_{\mathbf{k} + \varepsilon_2},$$

where $\varepsilon_1 := (1, 0)$ and $\varepsilon_2 := (0, 1)$. Clearly,

$$(1.1) \quad T_1 T_2 = T_2 T_1 \iff \beta_{\mathbf{k} + \varepsilon_1} \alpha_{\mathbf{k}} = \alpha_{\mathbf{k} + \varepsilon_2} \beta_{\mathbf{k}} \quad (\text{all } \mathbf{k} \in \mathbb{Z}_+^2).$$

In an entirely similar way one can define multivariable weighted shifts. Given $\mathbf{k} \in \mathbb{Z}_+^2$, the *moments* of (α, β) of order $\mathbf{k}$ is

$$\gamma_{\mathbf{k}} \equiv \gamma_{\mathbf{k}}(\alpha, \beta) := \begin{cases} 1 & \text{if } \mathbf{k} \equiv (k_1, k_2) = (0, 0) \\ \alpha_{(0,0)}^2 \cdots \alpha_{(k_1-1,0)}^2 & \text{if } k_1 \geq 1 \text{ and } k_2 = 0 \\ \beta_{(0,0)}^2 \cdots \beta_{(0,k_2-1)}^2 & \text{if } k_1 = 0 \text{ and } k_2 \geq 1 \\ \alpha_{(0,0)}^2 \cdots \alpha_{(k_1-1,0)}^2 \cdot \beta_{(k_1,0)}^2 \cdots \beta_{(k_1,k_2-1)}^2 & \text{if } k_1 \geq 1 \text{ and } k_2 \geq 1. \end{cases}$$

We remark that, due to the commutativity condition (1.1), $\gamma_{\mathbf{k}}$ can be computed using any nondecreasing path from $(0, 0)$ to (k_1, k_2) . We then recall basic results for 2-variable weighted shifts. The following is a criterion on hyponormality for 2-variable weighted shifts.

LEMMA 2. ([3])(*Six-point Test* (see Figure 1-(i))) Let $\mathbf{T} \equiv (T_1, T_2)$ be a 2-variable weighted shift, with weight sequences α and β . Then

$$\begin{aligned} [\mathbf{T}^*, \mathbf{T}] \geq 0 &\iff (([T_j^*, T_i] e_{\mathbf{k} + \varepsilon_j}, e_{\mathbf{k} + \varepsilon_i}))_{i,j=1}^2 \geq 0 \quad (\text{all } \mathbf{k} \in \mathbb{Z}_+^2) \\ &\iff \begin{pmatrix} \alpha_{\mathbf{k} + \varepsilon_1}^2 - \alpha_{\mathbf{k}}^2 & \alpha_{\mathbf{k} + \varepsilon_2} \beta_{\mathbf{k} + \varepsilon_1} - \alpha_{\mathbf{k}} \beta_{\mathbf{k}} \\ \alpha_{\mathbf{k} + \varepsilon_2} \beta_{\mathbf{k} + \varepsilon_1} - \alpha_{\mathbf{k}} \beta_{\mathbf{k}} & \beta_{\mathbf{k} + \varepsilon_2}^2 - \beta_{\mathbf{k}}^2 \end{pmatrix} \geq 0 \quad (\text{all } \mathbf{k} \in \mathbb{Z}_+^2). \end{aligned}$$

We now recall a well known characterization of subnormality for multivariable weighted shifts [14], due to C. Berger (cf. [2, II.6.10]) and independently established by Gellar and Wallen [13] in the single variable case:

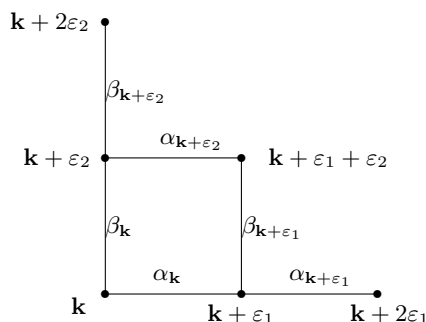


FIGURE 1. Weight diagram used in the Six-point Test

THEOREM 3 (Berger's Theorem). $W_{\alpha,\beta} \equiv (T_1, T_2)$ is subnormal if and only if there is a probability measure μ (which we call the Berger measure of $W_{\alpha,\beta}$) defined on the 2-dimensional rectangle $R = [0, \|T_1\|^2] \times [0, \|T_2\|^2]$ such that

$$\gamma_{\mathbf{k}} = \int_R s^{k_1} t^{k_2} d\mu(s, t), \quad \text{for all } \mathbf{k} \equiv (k_1, k_2) \in \mathbb{Z}_+^2.$$

In this paper we consider the following problem.

PROBLEM 4. (2-variable Subnormal Completion Problem) Given $m \geq 0$ and a finite collection of pairs of positive numbers $\Omega_m \equiv \{(\alpha_{\mathbf{k}}, \beta_{\mathbf{k}})\}_{|\mathbf{k}| \leq m}$ satisfying (1.1) for all $|\mathbf{k}| \leq m$ (where $|\mathbf{k}| := k_1 + k_2$), find necessary and sufficient conditions to guarantee the existence of a subnormal 2-variable weighted shift whose initial weights are given by Ω_m .

Problem 4 is closely related to the truncated real moment problems.

PROBLEM 5. Given real numbers

$$(1.2) \quad \gamma \equiv \gamma^{(2n)} := \gamma_{00}, \gamma_{10}, \gamma_{01}, \gamma_{20}, \gamma_{11}, \gamma_{02}, \dots, \gamma_{2n,0}, \dots, \gamma_{0,2n}$$

with $\gamma_{00} > 0$, the truncated real moment problem for γ entails finding conditions for the existence of a positive Borel measure μ , supported in $\mathbb{R}^2$, such that

$$\gamma_{ij} = \int x^i y^j d\mu, \quad 0 \leq i + j \leq n.$$

Given the truncated moment sequence γ in (2) and $0 \leq i, j \leq n$, we define the $(i+1) \times (j+1)$ matrix $M[i, j]$ whose entries are the moments of order $i+j$:

$$M[i, j](\gamma) \equiv M[i, j] := \begin{pmatrix} \gamma_{i+j,0} & \gamma_{i+j-1,1} & \cdots & \gamma_{i,j} \\ \gamma_{i+j-1,1} & \gamma_{i+j-2,2} & \cdots & \gamma_{i-1,j+1} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{j,i} & \gamma_{j-1,i+1} & \cdots & \gamma_{0,i+j} \end{pmatrix},$$

where we note that $M[i, j]$ has the Hankel-like property of being constant on each skew-diagonal; in particular, $M[i, i]$ is a self-adjoint Hankel matrix. We now define the *moment matrix* $M(n) \equiv M(n)(\gamma)$ via the block decomposition $M(n) := (M[i, j])_{0 \leq i, j \leq n}$. For example, if $n = 1$, the *quadratic moment problem* for $\gamma : \gamma_{00}, \gamma_{10}, \gamma_{01}, \gamma_{20}, \gamma_{11}, \gamma_{02}$ corresponds to

$$M(1) = \begin{pmatrix} M[0, 0] & M[0, 1] \\ M[1, 0] & M[1, 1] \end{pmatrix} = \begin{pmatrix} \gamma_{00} & \gamma_{10} & \gamma_{01} \\ \gamma_{10} & \gamma_{20} & \gamma_{11} \\ \gamma_{01} & \gamma_{11} & \gamma_{02} \end{pmatrix}$$

and if $n = 2$, the *quartic moment problem* for

$$\gamma : \gamma_{00}, \gamma_{10}, \gamma_{01}, \gamma_{20}, \gamma_{11}, \gamma_{02}, \gamma_{30}, \gamma_{21}, \gamma_{12}, \gamma_{03}, \gamma_{40}, \gamma_{31}, \gamma_{22}, \gamma_{13}, \gamma_{04}$$

corresponds to

$$\begin{aligned} M(2) &= \begin{pmatrix} M[0, 0] & M[0, 1] & M[0, 2] \\ M[1, 0] & M[1, 1] & M[1, 2] \\ M[2, 0] & M[2, 1] & M[2, 2] \end{pmatrix} \\ &= \begin{pmatrix} \gamma_{00} & \gamma_{10} & \gamma_{01} & \gamma_{20} & \gamma_{11} & \gamma_{02} \\ \gamma_{10} & \gamma_{20} & \gamma_{11} & \gamma_{30} & \gamma_{21} & \gamma_{12} \\ \gamma_{01} & \gamma_{11} & \gamma_{02} & \gamma_{21} & \gamma_{12} & \gamma_{03} \\ \gamma_{20} & \gamma_{30} & \gamma_{21} & \gamma_{40} & \gamma_{31} & \gamma_{22} \\ \gamma_{11} & \gamma_{21} & \gamma_{12} & \gamma_{31} & \gamma_{22} & \gamma_{13} \\ \gamma_{02} & \gamma_{12} & \gamma_{03} & \gamma_{22} & \gamma_{13} & \gamma_{04} \end{pmatrix}. \end{aligned}$$

(For basic results about truncated moment problems we refer to [4] and [6].)

We conclude this section some notations. For $n \geq 1$, let $m \equiv m(n) := \frac{(n+1)(n+2)}{2}$. For $A \in M_m(\mathbb{R})$ (the set of $m \times m$ real matrices), we denote the successive rows and columns according to the following lexicographic- functional ordering :

$$\mathbf{1}, X, Y, X^2, XY, Y^2, \dots, X^n, \dots, Y^n.$$

For A a positive $N \times N$ matrix, if $1 \leq n_1 < \dots < n_k \leq N$ we let

$$[A]_{\{n_1, \dots, n_k\}}$$

denote the compression of A to the rows and columns indexed by $\{n_1, \dots, n_k\}$. We let $\mathcal{C}_{\{n_1, \dots, n_k\}}$ and $\mathcal{R}_{\{n_1, \dots, n_k\}}$ denote the column space and the row space, respectively of A , i.e., the subspaces of $\mathbb{R}^N$ spanned by the columns and the rows indexed by $\{n_1, \dots, n_k\}$ of A , respectively. We also denote the entry of $A \in M_{m(n)}(\mathbb{R})$ in row $X^k Y^l$ and column $X^i Y^j$ by $A_{(k,l)(i,j)}$. Then it is easy to see that $M(n)_{(k,l)(i,j)} = \gamma_{k+i, l+j}$.

2. Main results

We start this section recalling the main results in [10]. In [10], it was shown:

THEOREM 6. [10] $\mathbf{T} \equiv (T_1, T_2)$ is subnormal if and only if $\mathbf{T} \equiv (T_1, T_2)$ is k -hyponormal for all $k \geq 1$.

THEOREM 7. [10] For a 2-variable weighted shift $W_{\alpha, \beta}$, the following are equivalent:

- (a) $W_{\alpha, \beta}$ is k -hyponormal;
- (b) $(\gamma_{\mathbf{k}+(n,m)+(p,q)} - \gamma_{\mathbf{k}+(n,m)}\gamma_{\mathbf{k}+(p,q)})_{\substack{1 \leq n+m \leq k \\ 1 \leq p+q \leq k}} \geq 0, \quad \forall \mathbf{k} \in \mathbb{Z}_+^2;$
- (c) $M_{\mathbf{k}}(k) := (\gamma_{\mathbf{k}+(n,m)+(p,q)})_{\substack{0 \leq n+m \leq k \\ 0 \leq p+q \leq k}} \geq 0, \quad \forall \mathbf{k} \in \mathbb{Z}_+^2.$

We thus obtain:

COROLLARY 8. $W_{\alpha, \beta}$ is subnormal if and only if

$$M_{\mathbf{k}}(\infty) := (\gamma_{\mathbf{k}+(m,n)+(p,q)})_{\substack{m+n \geq 0 \\ p+q \geq 0}} \geq 0$$

for $\mathbf{k} = (0, 0), (1, 0), (0, 1), (1, 1)$.

Proof. From Theorems 6 and 7, we can see that $W_{\alpha, \beta}$ is subnormal if and only if $M_{\mathbf{k}}(\infty) := (\gamma_{\mathbf{k}+(m,n)+(p,q)})_{\substack{m+n \geq 0 \\ p+q \geq 0}} \geq 0$ for all $\mathbf{k} \in \mathbb{Z}_+^2$. But since $M_{\mathbf{u}}(\infty)$ is a principal submatrix of one of $M_{\mathbf{k}}(\infty)$ for $\mathbf{k} = (0, 0), (1, 0), (0, 1), (1, 1)$, it follows from well-known result in the matrix analysis which states that a principal submatrix of a positive matrix is positive. $\square$

Corollary 8 provide a solution of the Stieltjes full power moment problem. Recall one variable Stieltjes full power moment problem. In 1894 Thomas Jan Stieltjes (1856-1894) published an extremely influential paper: *Recherches sur les fractions continues*, Ann. Fac. Sci. Toulouse, 8, 1-122; 9, 5-47. He introduced what is now known as the Stieltjes integral with respect to an increasing function ϕ , the latter describing

a distribution of mass (a measure μ) via the convention that the mass in an interval $[a, b]$ is $\mu([a, b]) = \phi(b) - \phi(a)$. This integral was used to solve the following problem which he called the moment problem:

PROBLEM 9. Given $\gamma \equiv \{\gamma_n\}_{n=0}^\infty$, find necessary and sufficient conditions on γ for the existence of positive measure μ with $\text{supp}\mu \subset \mathbb{R}^+$ such that

$$(2.1) \quad \gamma_n = \int x^n d\mu(x) \quad (\forall n \geq 0).$$

The number γ_n is called the n th moment of μ , and the sequence $\{\gamma_n\}$ is called the moment sequence of μ . Stieltjes was led to the Stieltjes moment problem above via a study of continued fractions. But now we can give an answer to the Problem 9 using Theorem 1.

PROPOSITION 10. The following statements are equivalent:

(i) There exists positive measure μ on $\mathbb{R}^+$ such that $\gamma_n = \int x^n d\mu(x)$ ($\forall n \geq 0$).

(ii) $\Lambda(p) \geq 0$ for all $p \geq 0$ on $\mathbb{R}^+$ where $\Lambda(p) := \sum_{i=0}^n a_i \gamma_i$ for $p(x) = \sum_{i=0}^n a_i x^i$.

(iii) $H(\infty) := \begin{pmatrix} \gamma_0 & \gamma_1 & \cdots \\ \gamma_1 & \gamma_2 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} \geq 0$, $H_x(\infty) := \begin{pmatrix} \gamma_1 & \gamma_2 & \cdots \\ \gamma_2 & \gamma_3 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} \geq 0$.

Proof. (i) $\implies$ (ii): Suppose there exist a positive measure μ with $\text{supp}\mu \subset \mathbb{R}_+$ satisfying (2.1). Then we have

$$\Lambda(p) = \sum_i a_i \gamma_i = \sum_i a_i \int x^i d\mu(x) = \int p(x) d\mu(x) \geq 0$$

for all $p(x) \geq 0$ on $\mathbb{R}_+$.

(ii) $\implies$ (iii): Suppose $\Lambda(p) \geq 0$ for all $p \geq 0$ on $\mathbb{R}^+$. Let $p(x) := |a_0 + a_1x + \cdots + a_nx^n|^2$ and $q(x) := x|a_0 + a_1x + \cdots + a_nx^n|^2$. Then $p(x) \geq 0$ and $q(x) \geq 0$ on $\mathbb{R}^+$. Thus, $\Lambda(p) = \sum_{i,j} a_i \bar{a}_j \gamma_{i+j} \geq 0$ and $\Lambda(q) = \sum_{i,j} a_i \bar{a}_j \gamma_{i+j+1} \geq 0$. Since n and a_i are arbitrary, it follows that $H(\infty) \geq 0$ and $H_x(\infty) \geq 0$.

(iii) $\implies$ (i): Suppose $H(\infty) \geq 0$ and $H_x(\infty) \geq 0$. Since $H(\infty) \geq 0$, we can see that $\gamma_0 > 0$. Let $\gamma' \equiv \{\gamma'_n\}_{n=0}^\infty$ with $\gamma'_n := \frac{\gamma_n}{\gamma_0}$. Then $H(\infty) \geq 0$ and $H_x(\infty) \geq 0$ for γ' . Now we can reproduce the weight sequence α from γ' via $\alpha_n := \sqrt{\frac{\gamma'_{n+1}}{\gamma'_n}}$. From Theorem 1, we can see that W_α is subnormal. Hence γ' has a Berger measure ν . Therefore $\mu = \gamma_0 \nu$. $\square$

The two-variable Stieltjes power moment problem states that:

PROBLEM 11. Given $\gamma \equiv \{\gamma_{\mathbf{k}}\}_{\mathbf{k} \in \mathbb{Z}_+^2}$, find necessary and sufficient conditions on γ for the existence of positive measure μ with $\text{supp } \mu \subset \mathbb{R}_+^2$ such that

$$\gamma_{\mathbf{k}} = \int x^{k_1} y^{k_2} d\mu(x, y) \quad (\forall \mathbf{k} := (k_1, k_2) \in \mathbb{Z}_+^2).$$

For given $\gamma \equiv \{\gamma_{\mathbf{k}}\}_{\mathbf{k} \in \mathbb{Z}_+^2}$, define a linear functional Γ_γ on $\mathbb{R}[x, y]$ by

$$\Gamma_\gamma(p) := \sum_{i,j} a_{ij} \gamma_{ij},$$

where $p(x, y) = \sum_{i,j} a_{ij} x^i y^j$. Then, from Corollary 8, we have an answer to the two-variable Stieltjes power moment problem.

PROPOSITION 12. Given $\gamma \equiv \{\gamma_{\mathbf{k}}\}_{\mathbf{k} \in \mathbb{Z}_+^2}$, the following statements are equivalent:

(i) There exists positive measure μ with $\text{supp } \mu \subset \mathbb{R}_+^2$ such that

$$(2.2) \quad \gamma_{\mathbf{k}} = \int x^{k_1} y^{k_2} d\mu(x, y) \quad (\forall \mathbf{k} := (k_1, k_2) \in \mathbb{Z}_+^2).$$

(ii) $\Gamma_\gamma(p) \geq 0$ for all $p(x, y) \geq 0$ on $\mathbb{R}_+^2$.

(iii) $M_{\mathbf{k}}(\infty) := (\gamma_{\mathbf{k}+(m,n)+(p,q)})_{\substack{m+n \geq 0 \\ p+q \geq 0}} \geq 0$ for $\mathbf{k} = (0, 0), (1, 0), (0, 1), (1, 1)$.

Proof. (i) $\implies$ (ii): Suppose there exist a positive measure μ with $\text{supp } \mu \subset \mathbb{R}_+^2$ satisfying (2.2). Then we have

$$\Gamma_\gamma(p) = \sum_{i,j} a_{ij} \gamma_{ij} = \sum_{i,j} a_{ij} \int x^i y^j d\mu(x, y) = \int p(x, y) d\mu(x, y) \geq 0$$

for all $p(x, y) \geq 0$ on $\mathbb{R}_+^2$.

(ii) $\implies$ (iii): Suppose $\Gamma_\gamma(p) \geq 0$ for all $p(x, y) \geq 0$ on $\mathbb{R}_+^2$. Let $p(x, y) := |a_{00} + a_{10}x + a_{01}y + \cdots + a_{n0}x^n + a_{n-1,1}x^{n-1}y + \cdots + a_{0n}y^n|^2$ and $q(x, y) := xp(x, y)$ and $r(x, y) := yp(x, y)$ and $s(x, y) := xyp(x, y)$. Then $p(x, y), q(x, y), r(x, y), s(x, y) \geq 0$ on $\mathbb{R}_+^2$. Thus, $\Gamma_\gamma(p) = \sum a_{ij} \overline{a_{kl}} \gamma_{i+k, j+l} \geq 0$, $\Gamma_\gamma(q) = \sum a_{ij} \overline{a_{kl}} \gamma_{i+k+1, j+l} \geq 0$, $\Gamma_\gamma(r) = \sum a_{ij} \overline{a_{kl}} \gamma_{i+k, j+l+1} \geq 0$ and $\Gamma_\gamma(s) = \sum a_{ij} \overline{a_{kl}} \gamma_{i+k+1, j+l+1} \geq 0$. Thus, it follows that $M_{\mathbf{k}}(\infty) \geq 0$ for $\mathbf{k} = (0, 0), (1, 0), (0, 1), (1, 1)$.

(iii) $\implies$ (i): If $M_{\mathbf{k}}(\infty) \geq 0$ for $\mathbf{k} = (0, 0), (1, 0), (0, 1), (1, 1)$, then by Corollary 8, we can construct a 2-variable weighted shift $W_{\alpha, \beta}$ which is subnormal. Thus, it has the Berger measure μ with $\text{supp } \mu \subset \mathbb{R}_+^2$ satisfying (2.2). $\square$

For the truncated moment problem, the direct analogue of the equivalence (i) $\iff$ (ii) above proposition need not hold. For example, let $\gamma : \gamma_{00} = \gamma_{10} = \gamma_{01} = \gamma_{20} = \gamma_{11} = 1$ and $\gamma_{02} = 2$. Then

- (1) $\Gamma_\gamma(p) \geq 0$ for all $p(x, y) \geq 0$ with $\deg p \leq 2$.
- (2) γ has no representing measure.
- (3) $M(1) = \begin{pmatrix} \gamma_{00} & \gamma_{10} & \gamma_{01} \\ \gamma_{10} & \gamma_{20} & \gamma_{11} \\ \gamma_{01} & \gamma_{11} & \gamma_{02} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix} \geq 0$.

Next we consider the main theorem in [6]. Although [6, Theorem 1.6] deals with truncated complex moment problems, there is an entirely equivalent version for the case of two real variables, which we now state. Let $K_{\mathcal{P}} := \{(x, y) \in \mathbb{R}^2 : p_i(x, y) \geq 0, \forall p_i \in \mathcal{P}\}$ for $\mathcal{P} \equiv \{p_1, \dots, p_m\} \subseteq \mathbb{R}[x, y]$ and define k_i by $\deg p_i = 2k_i$ or $\deg p_i = 2k_i - 1$ ($1 \leq i \leq m$).

In [6], Curto and Fialkow showed that the following statements are equivalent:

- (i) There exists a rank $M(n)$ -atomic representing measure μ for $\gamma^{(2n)}$ with $\text{supp} \mu \subseteq K_{\mathcal{P}}$.
- (ii) $M(n) \geq 0$ and there is some flat (i.e., rank-preserving) extension $M(n+1)$ for which $M_{p_i}(n+k_i) \geq 0$ ($1 \leq i \leq m$).

In this case, the representing measure for $M(n+1)$ is rank $M(n)$ -atomic, supported in $K_{\mathcal{P}}$, and with precisely $\text{rank} M(n) - \text{rank} M_{p_i}(n+k_i)$ atoms in $\mathcal{Z}(p_i) := \{(x, y) \in \mathbb{R}^2 : p_i(x, y) = 0\}$ ($1 \leq i \leq m$).

If we let $p_1(x, y) := x$ and $p_2(x, y) := y$, then $K_{\mathcal{P}} = \mathbb{R}_+^2$, $M_{p_1}(n+k_1) = M_x(n+1)$ and $M_{p_2}(n+k_2) = M_y(n+1)$. Thus, we have an abstract solution of Problem 4(Odd case):

THEOREM 13. *If m is odd, then the following statements are equivalent:*

- (i) Ω_m has a subnormal completion.
- (ii) There exists a rank $M(k)$ -atomic representing measure μ for $\gamma(\Omega_m)$ supported in $\mathbb{R}_+^2$ where $k := \lfloor \frac{m+1}{2} \rfloor$.
- (iii) $M(\Omega_m) = M(k) \geq 0$ and Ω_m admits a commutative extension $\widehat{\Omega}_{m+2}$ such that the moment matrix $M(\widehat{\Omega}_{m+2}) = M(k+1)$ is a flat extension of $M(k)$, $M_x(k+1) \geq 0$ and $M_y(k+1) \geq 0$.

In this case, the Berger measure μ of a subnormal completion $\widehat{\Omega}_\infty$ of Ω_m has $\text{rank} M(k) - \text{rank} M_x(k+1)$ atoms in $\{0\} \times \mathbb{R}_+$ (resp. $\text{rank} M(k) - \text{rank} M_y(k+1)$ atoms in $\mathbb{R}_+ \times \{0\}$). We remark that the Theorem 13 resembles the Corollary 8 in the sense that $M = M_{00}$, $M_x = M_{10}$, $M_y = M_{01}$.

In 1966, Stampfli [16] showed that $\alpha : \alpha_0 < \alpha_1 < \alpha_2$ has always subnormal completions, directly constructing the normal extension. It was highly nontrivial that time. But it is easy now from Theorem 1.

Similarly, we can give a concrete criterion of the Problem 4 for the case $m = 1$.

THEOREM 14. *Given $\Omega_1 := \{(\alpha_{\mathbf{k}}, \beta_{\mathbf{k}}) : |\mathbf{k}| \leq 1\}$ satisfying $\beta_{10}\alpha_{00} = \alpha_{01}\beta_{00}$, the following statement are equivalent:*

- (i) Ω_1 has a subnormal completion;
- (ii) Ω_1 has a hyponormal completion;
- (iii) $(\alpha_{10}^2 - \alpha_{00}^2)(\beta_{01}^2 - \beta_{00}^2) \geq (\alpha_{01}\beta_{10} - \alpha_{00}\beta_{00})^2$;
- (iv) $M(1) := \begin{pmatrix} \gamma_{00} & \gamma_{01} & \gamma_{10} \\ \gamma_{01} & \gamma_{02} & \gamma_{11} \\ \gamma_{10} & \gamma_{11} & \gamma_{20} \end{pmatrix} = \begin{pmatrix} 1 & \alpha_{00}^2 & \beta_{00}^2 \\ \alpha_{00}^2 & \alpha_{00}^2\alpha_{10}^2 & \alpha_{00}^2\beta_{10}^2 \\ \beta_{00}^2 & \alpha_{00}^2\beta_{10}^2 & \beta_{00}^2\beta_{01}^2 \end{pmatrix} \geq 0$.

Proof. See [12]. □

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