

## ON RETARDED INTEGRAL INEQUALITIES OF BIHARI-TYPE

SUNG KYU CHOI\*, TAEYOUNG CHOI\*\*, DAEJUNG KIM\*\*\*,  
AND NAMJIP KOO\*\*\*\*

ABSTRACT. We obtain some retarded integral inequalities of Bihari-type and apply these results to a retarded differential equation of Bernoulli-type.

### 1. Introduction

Differential and integral inequalities have played a fundamental role in the study of existence, uniqueness, boundedness, stability, invariant manifolds and other qualitative properties of solutions of differential equations and integral equations. This is the case of the attractive Gronwall-Bellman inequality [4]. Bihari's inequality is the most important nonlinear generalization of the Gronwall-Bellman inequality. Over the years, various investigators have discovered many useful integral inequalities in order to achieve a diversity of desired goals [1–4, 6, 8, 9, 11, 13–15].

Lipovan [7] gave a useful nonlinear generalization of the celebrated Gronwall inequality and presented some of its applications. However, the integral inequalities available in the literature do not apply directly in certain general situations. Pachpatte [12] gave explicit bounds on more general retarded integral inequalities which can be used as tools in the qualitative study of certain retarded integrodifferential equations.

---

Received December 28, 2008; Revised February 16, 2009; Accepted February 17, 2009.

2000 Mathematics Subject Classification: Primary 34A30, 34D23.

Key words and phrases: linear differential inequalities, retarded integral inequalities, retarded differential equations, retarded integrodifferential equations.

Correspondence should be addressed to Namjip Koo, njkoo@math.cnu.ac.kr

This work was supported by the R & E program of KOSEF in 2008.

In this paper we obtain some retarded integral inequalities of Bihari-type and apply these results to a retarded differential equation of Bernoulli-type. Furthermore, we give two examples to explain our results.

## 2. Main results

In this section we consider retarded linear integral inequalities of Gronwall type for scalar functions of one variable. These retarded inequalities can be used in the study of a retarded nonlinear differential equation of Bernoulli-type. We derive an estimate of a retarded linear integral inequality. Let  $\mathbb{R}^+ = [0, \infty)$  and  $I = [t_0, T)$  be the subsets of  $\mathbb{R}$ .

**THEOREM 2.1.** *Let  $u(t)$ ,  $a(t)$ , and  $b(t)$  be nonnegative continuous functions in  $I$  and  $\alpha(t) \in C^1(I)$  be nondecreasing with  $\alpha(t) \leq t$  on  $I$ . Suppose that*

$$u(t) \leq c + \int_{\alpha(t_0)}^{\alpha(t)} a(s)ds + \int_{\alpha(t_0)}^{\alpha(t)} b(s)u(s)ds, \quad t, t_0 \in I, \quad (2.1)$$

where  $c$  is a nonnegative constant. Then for  $t \geq t_0$ ,

$$\begin{aligned} u(t) &\leq c \exp \left( \int_{\alpha(t_0)}^{\alpha(t)} b(s)ds \right) \\ &+ \int_{\alpha(t_0)}^{\alpha(t)} a(s) \exp \left( \int_s^{\alpha(t)} b(\tau)d\tau \right) ds, \quad t \in I. \end{aligned} \quad (2.2)$$

*Proof.* Define  $v(t)$  by the right hand side of (2.1). Then  $u(t) \leq v(t)$  and  $v(t_0) = c$ . Then we obtain

$$\begin{aligned} v'(t) &\leq b(\alpha(t))v(\alpha(t))\alpha'(t) + a(\alpha(t))\alpha'(t) \\ &\leq b(\alpha(t))v(t)\alpha'(t) + a(\alpha(t))\alpha'(t), \quad t \in I \end{aligned} \quad (2.3)$$

by the monotonicity of  $v$  and  $\alpha(t) \leq t$ . The inequality (2.3) implies

$$\begin{aligned} &\frac{d}{ds} \left( v(s) \exp \left( \int_s^t b(\alpha(\tau))\alpha'(\tau)d\tau \right) \right) \\ &= [v'(s) - b(\alpha(s))v(s)\alpha'(s)] \exp \left( \int_s^t b(\alpha(\tau))\alpha'(\tau)d\tau \right) \\ &\leq a(\alpha(s))\alpha'(s) \exp \left( \int_s^t b(\alpha(\tau))\alpha'(\tau)d\tau \right), \quad t \geq s \geq t_0. \end{aligned}$$

Integration over  $s$  from  $t_0$  to  $t$  gives

$$\begin{aligned} v(t) &= v(t_0) \exp \left( \int_{t_0}^t b(\alpha(\tau)) \alpha'(\tau) d\tau \right) \\ &\leq \int_{t_0}^t a(\alpha(s)) \alpha'(s) \exp \left( \int_s^t b(\alpha(\tau)) \alpha'(\tau) d\tau \right) ds \end{aligned}$$

and the change of variables yields

$$\begin{aligned} u(t) &\leq v(t) \\ &\leq v(t_0) \exp \left( \int_{\alpha(t_0)}^{\alpha(t)} b(\tau) d\tau \right) + \int_{\alpha(t_0)}^{\alpha(t)} a(s) \exp \left( \int_s^{\alpha(t)} b(\tau) d\tau \right) ds. \end{aligned}$$

The proof is complete.  $\square$

Using the result of Theorem 2.1 we obtain the retarded linear differential inequality.

**COROLLARY 2.2.** *Let  $b(t)$  and  $a(t)$  be continuous functions for  $t \geq t_0$  and let  $v(t)$  and  $\alpha(t)$  be differentiable nonnegative functions for  $t \geq t_0$  with  $\alpha(t) \leq t$  and  $\alpha'(t) \neq 0$ . Suppose*

$$v'(t) \leq a(t) + b(t)v(\alpha(t)), t \in I.$$

Then for  $t \geq t_0$ ,

$$v(t) \leq v(t_0) \exp \left( \int_{\alpha(t_0)}^{\alpha(t)} \hat{b}(\tau) d\tau \right) + \int_{\alpha(t_0)}^{\alpha(t)} \hat{a}(s) \exp \left( \int_s^{\alpha(t)} \hat{b}(\tau) d\tau \right) ds,$$

where  $\hat{a}(t) = \frac{a(\alpha^{-1}(t))}{\alpha'(t)}$  and  $\hat{b} = \frac{b(\alpha^{-1}(t))}{\alpha'(t)}$ .

**REMARK 2.3.** Let  $\alpha(t) = t$  in Theorem 2.1. Then we obtain a comparison result for linear integral inequalities:

$$\begin{aligned} u(t) &\leq u(t_0) \exp \left( \int_{t_0}^t b(s) ds \right) \\ &+ \int_{t_0}^t a(s) \exp \left( \int_s^t b(\tau) d\tau \right) ds, \quad t \in I. \end{aligned} \tag{2.4}$$

Now, we investigate some retarded nonlinear integral inequalities. To this end we need the following lemmas in [12].

**LEMMA 2.4.** [12, Theorem 2.1] *Let  $u(t), a(t) \in C(I, \mathbb{R}^+)$ ,  $b(t, s) \in C(I \times I, \mathbb{R}^+)$  for  $t_0 \leq s \leq t \leq T$  and  $\alpha(t) \in C^1(I)$  be nondecreasing with*

$\alpha(t) \leq t$  on  $I$  and  $c$  be a positive constant. Suppose that  $w \in C((0, \infty))$  is a nondecreasing function. If

$$u(t) \leq c + \int_{\alpha(t_0)}^{\alpha(t)} \left( a(s)w(u(s)) + \int_{\alpha(t_0)}^s b(s, \sigma)w(u(\sigma))d\sigma \right) ds, \quad (2.5)$$

for  $t \in I$ , then for  $t_0 \leq t \leq b_1$ ,

$$u(t) \leq W^{-1} \left[ W(c) + \int_{\alpha(t_0)}^{\alpha(t)} \left( a(s) + \int_{\alpha(t_0)}^s b(s, \sigma)d\sigma \right) ds \right], \quad (2.6)$$

where  $W^{-1}$  is the inverse function of  $W(u) = \int_{u_0}^u \frac{dz}{w(z)}$ ,  $u > 0, u_0 > 0$ , and  $b_1$  is the largest number such that  $b_1 \geq t_0$  and for  $t_0 \leq t \leq b_1$

$$\left[ W(c) + \int_{\alpha(t_0)}^{\alpha(t)} \left( a(s) + \int_{\alpha(t_0)}^s b(s, \sigma)d\sigma \right) ds \right] \in \text{Dom}(W^{-1}).$$

REMARK 2.5. If we set  $\alpha(t) = t$  and  $b(t, s) = 0$  in [12, Theorem 2.1]. then we obtain the well-known Bihari-type inequality.

We extend the result in [14, Lemma 1] to a retarded nonlinear integral inequality.

THEOREM 2.6. Let  $u(t), a(t), b(t) \in C(I, \mathbb{R}^+)$  and  $\alpha(t) \in C^1(I)$  be nondecreasing with  $\alpha(t) \leq t$  on  $I$  and  $c$  be a positive constant. Let  $w_i, i = 1, 2$ , be continuous nondecreasing functions on  $\mathbb{R}^+$  and positive on  $(0, \infty)$  such that  $w = \frac{w_2}{w_1}$  is nondecreasing on  $(0, \infty)$ . If

$$u(t) \leq c + \int_{\alpha(t_0)}^{\alpha(t)} a(s)w_1(u(s))ds + \int_{\alpha(t_0)}^{\alpha(t)} b(s)w_2(u(s))ds, \quad (2.7)$$

for  $t \in I$ , then for  $t_0 \leq t \leq b_1$ ,

$$u(t) \leq W_2^{-1} \left[ W_2(c_1) + \int_{\alpha(t_0)}^{\alpha(t)} b(s)ds \right], \quad t_0 \leq t \leq b_1,$$

where  $W_1(u) = \int_c^u \frac{dz}{w_1(z)}$ ,  $u > 0$ ,  $W_2(u) = \int_{u_0}^u \frac{dz}{w_2(z)}$ ,  $u_0 > 0$ , and  $b_1$  is the largest number such that  $b_1 \geq t_0$  and

$$\begin{aligned} |a|_{b_1} &= \int_{t_0}^{b_1} a(s)ds \leq \int_c^\infty \frac{dz}{w_1(z)}, \\ |b|_{b_1} &= \int_{t_0}^{b_1} b(s)ds \leq \int_{c_1}^\infty \frac{dz}{w_2(z)}, \quad c_1 = W_1^{-1}(|a|_{b_1}). \end{aligned}$$

*Proof.* Let  $v(t)$  by the right side of (2.7). Then  $v(t_0) = c, u(t) \leq v(t)$  and we obtain

$$\begin{aligned} v'(t) &\leq [a(\alpha(t))w_1(v(\alpha(t))) + b(\alpha(t))w_2(v(\alpha(t)))] \alpha'(t) \\ &\leq w_1(v(t)) \left[ a(\alpha(t)) + \frac{w_2(v(\alpha(t)))}{w_1(v(\alpha(t)))} b(\alpha(t)) \right] \alpha'(t), \end{aligned}$$

by the monotonicity of  $w_1$  and  $\alpha(t) \leq t$ . This implies

$$\begin{aligned} \frac{d(W_1(v(t)))}{dt} &= \frac{v'(t)}{w_1(v(t))} \\ &\leq [a(\alpha(t)) + w(v(\alpha(t)))b(\alpha(t))] \alpha'(t). \end{aligned} \quad (2.8)$$

Integrating (2.8) from  $t_0$  to  $t$  and making the change of variables, we have

$$\begin{aligned} W_1(v(t)) &= W_1(v(t_0)) + \int_{t_0}^t [a(\alpha(\sigma)) + w(v(\alpha(\sigma)))b(\alpha(\sigma))] \alpha'(\sigma) d\sigma \\ &= W_1(c) + \int_{\alpha(t_0)}^{\alpha(t)} [a(s) + b(s)w(v(s))] ds \\ &\leq \tilde{c} + \int_{\alpha(t_0)}^{\alpha(t)} b(s)w(v(s)) ds, \end{aligned}$$

where  $\tilde{c} = \int_{\alpha(t_0)}^{b_1} a(s) ds$ . Then, putting  $z(t) = W_1(v(t))$ , we obtain

$$z(t) \leq \tilde{c} + \int_{\alpha(t_0)}^{\alpha(t)} b(s)w(W_1^{-1}(z(s))) ds.$$

Setting  $b(t, s) = 0$  in [12, Theorem 2.1], we get, for  $t \in [t_0, b_1)$

$$z(t) \leq \tilde{W}^{-1} \left[ \tilde{W}(\tilde{c}) + \int_{\alpha(t_0)}^{\alpha(t)} b(s) ds \right],$$

where  $\tilde{W}(u) = \int_{\tilde{u}_0}^u \frac{ds}{w(W_1^{-1}(s))}$ ,  $\tilde{u}_0 = W_1(u_0)$ . Since  $u \leq v = W_1^{-1}(z)$  and  $\tilde{W} = W_2 \circ W_1^{-1}$ , we have

$$u(t) \leq W_2^{-1} \left[ W_2(c_1) + \int_{\alpha(t_0)}^{\alpha(t)} b(s) ds \right], \quad t_0 \leq t \leq b_1.$$

The proof is complete.  $\square$

We obtain the retarded nonlinear integral inequality of Bernoulli-type by using the consequences of Theorem 2.6.

COROLLARY 2.7. Let  $w_1(u) = u$  and  $w_2(u) = u^p$  with  $0 < p \neq 1$  constant in Theorem 2.6. Let  $u(t), a(t)$ , and  $b(t)$  be nonnegative continuous functions in  $I$ , and  $c > 0$  be constant. Suppose that

$$u(t) \leq c + \int_{\alpha(t_0)}^{\alpha(t)} a(s)u(s)ds + \int_{\alpha(t_0)}^{\alpha(t)} b(s)u^p(s)ds, \quad (2.9)$$

holds for  $t \geq t_0$ . Then for  $0 \leq t_0 \leq t \leq b_1$ ,

$$\begin{aligned} u(t) &\leq \begin{cases} W_1^{-1} \left[ W_1(c_2) + \int_{\alpha(t_0)}^{\alpha(t)} a(s)ds \right], & 0 < p < 1 \\ W_2^{-1} \left[ W_2(c_1) + \int_{\alpha(t_0)}^{\alpha(t)} b(s)ds \right], & 1 < p \end{cases} \\ &= \begin{cases} c_2 \exp \left( \int_{\alpha(t_0)}^{\alpha(t)} a(s)ds \right), & 0 < p < 1 \\ \left[ c_1^{1-p} + (1-p) \int_{\alpha(t_0)}^{\alpha(t)} b(s)ds \right]^{\frac{1}{1-p}}, & 1 < p, \end{cases} \end{aligned} \quad (2.10)$$

where

$$\begin{aligned} c_1 &= W_1^{-1}(|a|_{b_1}) = c \exp(|a|_{b_1}), \\ c_2 &= W_2^{-1}(|b|_{b_1}) = [c^{1-p} + (1-p)|b|_{b_1}]^{\frac{1}{1-p}}. \end{aligned}$$

*Proof.* It is easy to find that

$$\begin{aligned} W_1(u) &= \ln\left(\frac{u}{c}\right), \quad W_1^{-1}(u) = c \exp u, \\ W_2(u) &= \frac{1}{1-p} [u^{1-p} - c^{1-p}], \quad W_2^{-1}(u) = [c^{1-p} + (1-p)u]^{\frac{1}{1-p}}. \end{aligned}$$

We obtain the increasing function  $w$  on  $(0, \infty)$  which can be applied in Theorem 2.6:

$$w(u) = \begin{cases} \frac{w_1(u)}{w_2(u)} = u^{1-p}, & 0 < p < 1 \\ \frac{w_2(u)}{w_1(u)} = u^{p-1}, & 1 < p. \end{cases}$$

Applying this to (2.9) in Theorem 2.6, we obtain the inequality (2.10).  $\square$

REMARK 2.8. For the cases  $p = 0$  and  $p = 1$  we obtain the retarded linear integral inequality (2.1) in Theorem 2.1.

If the condition on  $p$  in Corollary 2.7 is replaced by  $0 < p < 1$ , then we obtain more concrete inequality than the inequality in Corollary 2.7 by

using the consequence of retarded integral inequality in Theorem 2.1. We extend the result of Theorem 2.1 to the following corollary.

**COROLLARY 2.9.** *Let  $w_1(u) = u$  and  $w_2(u) = u^p$  with  $0 < p < 1$  in Theorem 2.6. Let  $u(t)$ ,  $a(t)$ , and  $b(t)$  be nonnegative continuous functions in  $I$ , and  $c > 0$  be a constant. Suppose that*

$$u(t) \leq c + \int_{\alpha(t_0)}^{\alpha(t)} a(s)u(s)ds + \int_{\alpha(t_0)}^{\alpha(t)} b(s)u^p(s)ds \quad (2.11)$$

holds for  $t \in I$ . Then for  $0 \leq t_0 \leq t$ ,

$$u(t) \leq \left[ \left( c \exp\left( \int_{\alpha(t_0)}^{\alpha(t)} a(s)ds \right) \right)^q + q \int_{\alpha(t_0)}^{\alpha(t)} b(s) \exp\left( q \int_s^{\alpha(t)} a(\tau)d\tau \right) ds \right]^{\frac{1}{q}},$$

where  $q = 1 - p$ .

*Proof.* Define  $v(t)$  by the right hand side of (2.11). Then  $u(t) \leq v(t)$  and  $v(t_0) = c$ . Then we obtain

$$\begin{aligned} v'(t) &\leq v^p(\alpha(t)) [b(\alpha(t)) + a(\alpha(t))v^{1-p}(\alpha(t))] \alpha'(t) \\ &\leq v^p(t) [b(\alpha(t)) + a(\alpha(t))v^{1-p}(\alpha(t))] \alpha'(t), \quad t \in I \end{aligned} \quad (2.12)$$

by the monotonicity of  $v$  and  $\alpha(t) \leq t$ , which implies

$$\frac{v'(t)}{v^p(t)} \leq [b(\alpha(t)) + a(\alpha(t))v^{1-p}(\alpha(t))] \alpha'(t), \quad t \in I. \quad (2.13)$$

By integrating (2.13) from  $t_0$  to  $t$  and the change of variables we have

$$\frac{1}{q} v^q(t) \leq \frac{c^q}{q} + \int_{\alpha(t_0)}^{\alpha(t)} b(s)ds + \int_{\alpha(t_0)}^{\alpha(t)} a(s)v^q(s)ds,$$

where  $q = 1 - p$ . Letting  $z(t) = \frac{1}{q} v^q(t)$ , we obtain

$$z(t) \leq \frac{c^q}{q} + \int_{\alpha(t_0)}^{\alpha(t)} b(s)ds + q \int_{\alpha(t_0)}^{\alpha(t)} a(s)z(s)ds,$$

which implies

$$z(t) \leq \frac{c^q}{q} \exp\left( q \int_{\alpha(t_0)}^{\alpha(t)} a(s)ds \right) + \int_{\alpha(t_0)}^{\alpha(t)} b(s) \exp\left( q \int_s^{\alpha(t)} a(\tau)d\tau \right) ds,$$

by Theorem 2.1. Hence we have

$$\begin{aligned} u(t) &\leq v(t) \\ &\leq \left[ c^q \exp\left( q \int_{\alpha(t_0)}^{\alpha(t)} a(s)ds \right) + q \int_{\alpha(t_0)}^{\alpha(t)} b(s) \exp\left( q \int_s^{\alpha(t)} a(\tau)d\tau \right) ds \right]^{\frac{1}{q}}, \end{aligned}$$

where  $q = 1 - p$ . The proof is complete.  $\square$

REMARK 2.10. If we set  $p = 0$  in Corollary 2.9, then we obtain the same retarded integral inequality as in Theorem 2.1. Also, if we set  $\alpha(t) = t$  in Corollary 2.9 we obtain the same result as in [3, Lemma 4.1].

COROLLARY 2.11. *Let  $v(t)$  be a positive differentiable function satisfying the differential inequality*

$$v'(t) \leq a(t)v(\alpha(t)) + b(t)v^p(\alpha(t)), \quad t \in I, \quad (2.14)$$

where  $a(t)$  and  $b(t)$  are nonnegative continuous functions in  $I$ , and  $0 < p \neq 1$  is a constant. Then for  $0 \leq t_0 \leq t \leq b_1$ ,

$$\begin{aligned} v(t) &\leq \begin{cases} W_1^{-1} \left[ W_1(c_2) + \int_{\alpha(t_0)}^{\alpha(t)} \hat{a}(s) ds \right], & 0 < p < 1 \\ W_2^{-1} \left[ W_2(c_1) + \int_{\alpha(t_0)}^{\alpha(t)} \hat{b}(s) ds \right], & 1 < p \end{cases} \\ &= \begin{cases} c_2 \exp \left( \int_{\alpha(t_0)}^{\alpha(t)} \hat{a}(s) ds \right), & 0 < p < 1 \\ \left[ c_1^q + q \int_{\alpha(t_0)}^{\alpha(t)} \hat{b}(s) ds \right]^{\frac{1}{q}}, & 1 < p, \end{cases} \end{aligned}$$

where  $q = 1 - p$ ,  $\hat{a}(t) = \frac{a(\alpha^{-1}(t))}{\alpha'(t)}$ ,  $\hat{b}(t) = \frac{b(\alpha^{-1}(t))}{\alpha'(t)}$ ,  $c_1 = W_1^{-1}(|a|_{b_1}) = c \exp(|a|_{b_1})$ , and  $c_2 = W_2^{-1}(|b|_{b_1}) = [c^q + q|b|_{b_1}]^{\frac{1}{q}}$ .

*Proof.* The integration of (2.14) and the change of variables yield

$$v(t) \leq v(t_0) + \int_{\alpha(t_0)}^{\alpha(t)} \hat{a}(s)v(s)ds + \int_{\alpha(t_0)}^{\alpha(t)} \hat{b}(s)v^p(s)ds, \quad t \in I,$$

where  $\hat{a}(t) = \frac{a(\alpha^{-1}(t))}{\alpha'(t)}$  and  $\hat{b}(t) = \frac{b(\alpha^{-1}(t))}{\alpha'(t)}$ . By applying this to Corollary 2.7 we obtain the required estimate.  $\square$

REMARK 2.12. If we set  $\alpha(t) = t$  in Theorem 2.6, then we obtain the same result as in [5, Lemma 2.1].

We illustrate our results by the following example, including a retarded differential equation of Bernoulli-type.

EXAMPLE 2.13. *Consider the retarded differential equation with the given initial condition*

$$x'(t) = a(t)x(t - h(t)) + b(t)x^p(t - h(t)), \quad x(t_0) = x_0 \quad (2.15)$$

where  $h(t) \in C^1(I)$  is nondecreasing with  $t - h(t) \geq 0$ ,  $h'(t) < 1$ ,  $h(t_0) = 0$  and  $p \neq 1$  is a positive constant. Then the solution  $x(t)$  of (2.15) satisfies

$$|x(t)| \leq \begin{cases} c_2 \exp \left( \int_{t_0}^{t-h(t)} \hat{a}(s) ds \right), & 0 < p < 1 \\ \left[ c_1^{1-p} + (1-p) \left( \int_{t_0}^{t-h(t)} \hat{b}(s) ds \right) \right]^{\frac{1}{1-p}}, & 1 < p, \end{cases} \quad (2.16)$$

where  $\hat{a}(t) = \frac{|a(\alpha^{-1}(t))|}{|1-h'(\alpha^{-1}(t))|}$ ,  $\hat{b}(t) = \frac{|b(\alpha^{-1}(t))|}{|1-h'(\alpha^{-1}(t))|}$  and  $\alpha(t) = t - h(t)$ .

*Proof.* We see that the solution  $x(t)$  of (2.15) satisfies the integral equation

$$\begin{aligned} x(t) &= x(t_0) + \int_{\alpha(t_0)}^{\alpha(t)} a(\alpha^{-1}(s))x(s) \frac{ds}{1-h'(\alpha^{-1}(s))} \\ &\quad + \int_{\alpha(t_0)}^{\alpha(t)} b(\alpha^{-1}(s))x^p(s) \frac{ds}{1-h'(\alpha^{-1}(s))}, \end{aligned}$$

where  $\alpha^{-1}(t)$  is the inverse function of  $\alpha(t) = t - h(t)$ , by applying the change of variables. Putting  $|x(t)| = u(t)$ , we have

$$u(t) \leq u(t_0) + \int_{\alpha(t_0)}^{\alpha(t)} \hat{a}(s)u(s)ds + \int_{\alpha(t_0)}^{\alpha(t)} \hat{b}(s)u^p(s)ds, \quad (2.17)$$

where  $\hat{a}(t) = \frac{|a(\alpha^{-1}(t))|}{|1-h'(\alpha^{-1}(t))|}$ ,  $\hat{b}(t) = \frac{|b(\alpha^{-1}(t))|}{|1-h'(\alpha^{-1}(t))|}$  and  $\alpha(t) = t - h(t)$ . A suitable application of the inequality given in Corollary 2.7 to (2.17) yields the required estimate in (2.16).  $\square$

**THEOREM 2.14.** Let  $u(t), a(t) \in C(I, \mathbb{R}^+)$ ,  $b(t, s) \in C(I \times I, \mathbb{R}^+)$  for  $t_0 \leq s \leq t \leq T$  and  $\alpha(t) \in C^1(I)$  be nondecreasing with  $\alpha(t) \leq t$  on  $I$  and  $c$  be a positive constant. Let  $w_i, i = 1, 2$ , be continuous nondecreasing functions on  $\mathbb{R}^+$  and positive on  $(0, \infty)$  such that  $w = \frac{w_2}{w_1}$  is nondecreasing on  $(0, \infty)$ . If

$$u(t) \leq c + \int_{\alpha(t_0)}^{\alpha(t)} \left[ a(s)w_1(u(s)) + \int_{\alpha(t_0)}^s b(s, \sigma)w_2(u(\sigma))d\sigma \right] ds, \quad (2.18)$$

for  $t \in I$ , then for  $t_0 \leq t \leq b_1$ ,

$$u(t) \leq W_2^{-1} [W_2(c_1) + A(t)], \quad (2.19)$$

where  $A(t) = \int_{\alpha(t_0)}^{\alpha(t)} [\int_{\alpha(t_0)}^s b(s, \sigma) d\sigma] ds$ ,  $W_1(u) = \int_c^u \frac{dz}{w_1(z)}$ ,  $u > 0$ ,  $W_2(u) = \int_{u_0}^u \frac{dz}{w_2(z)}$ ,  $u_0 > 0$  and  $b_1$  is the largest number such that  $b_1 \geq t_0$  and

$$\begin{aligned} |a|_{b_1} &= \int_{t_0}^{b_1} a(s) ds \leq \int_c^\infty \frac{dz}{w_1(z)}, \\ |A|_{b_1} &= \int_{t_0}^{b_1} A(s) ds \leq \int_{c_1}^\infty \frac{dz}{w_2(z)}, c_1 = W_1^{-1}(|a|_{b_1}). \end{aligned}$$

*Proof.* Let  $v(t)$  by the right side of (2.18). Then  $v(t_0) = c$ ,  $u(t) \leq v(t)$  and we obtain

$$\begin{aligned} v'(t) &\leq \left[ a(\alpha(t))w_1(v(\alpha(t))) + \int_{\alpha(t_0)}^{\alpha(t)} b(\alpha(t), s)w_2(v(\alpha(s)))ds \right] \alpha'(t) \\ &\leq \left[ a(\alpha(t))w_1(v(t)) + w_2(v(t)) \int_{\alpha(t_0)}^{\alpha(t)} b(\alpha(t), s)ds \right] \alpha'(t) \\ &\leq w_1(v(t)) \left[ a(\alpha(t)) + \frac{w_2(v(\alpha(t)))}{w_1(v(\alpha(t)))} \int_{\alpha(t_0)}^{\alpha(t)} b(\alpha(t), s)ds \right] \alpha'(t), \end{aligned}$$

in view of the monotonicity of  $w_1$  and  $\alpha(t) \leq t$ . It follows that

$$\begin{aligned} \frac{d(W_1(v(t)))}{dt} &= \frac{v'(t)}{w_1(v(t))} \\ &\leq \left[ a(\alpha(t)) + w(v(\alpha(t))) \int_{\alpha(t_0)}^{\alpha(t)} b(\alpha(t), s)ds \right] \alpha'(t). \end{aligned} \tag{2.20}$$

Integrating (2.20) from  $t_0$  to  $t$  and making the change of variables, we have

$$\begin{aligned} W_1(v(t)) &= W_1(v(t_0)) \\ &+ \int_{t_0}^t \left[ a(\alpha(\sigma)) + w(v(\alpha(\sigma))) \int_{\alpha(t_0)}^{\alpha(\sigma)} b(\alpha(\sigma), \tau) d\tau \right] \alpha'(\sigma) d\sigma \\ &= W_1(c) + \int_{\alpha(t_0)}^{\alpha(t)} \left[ a(s) + w(v(s)) \int_{t_0}^s b(s, \tau) d\tau \right] ds \\ &\leq \tilde{c} + \int_{\alpha(t_0)}^{\alpha(t)} \left[ \int_{t_0}^s b(s, \tau) d\tau \right] w(v(s)) ds, \end{aligned}$$

where  $\tilde{c} = \int_{\alpha(t_0)}^{b_1} a(s)ds$ . Then, putting  $z(t) = W_1(v(t))$ , we obtain

$$z(t) \leq \tilde{c} + \int_{\alpha(t_0)}^{\alpha(t)} \left[ \int_{t_0}^s b(s, \tau) d\tau \right] w(W_1^{-1}(z(t))) ds.$$

By Lemma 2.4, we get, for  $t \in [t_0, b_1)$

$$z(t) \leq \tilde{W}^{-1} \left[ \tilde{W}(\tilde{c}) + \int_{\alpha(t_0)}^{\alpha(t)} \left( \int_{t_0}^s b(s, \tau) d\tau \right) ds \right],$$

where  $\tilde{W}(u) = \int_{u_0}^u \frac{ds}{w(W_1^{-1}(s))}$  and  $u_0 = W_1(u_0)$ . Since  $u \leq v = W_1^{-1}(z)$  and  $\tilde{W} = W_2 \circ W_1^{-1}$ , we have

$$u(t) \leq W_2^{-1} \left[ W_2(c_1) + \int_{\alpha(t_0)}^{\alpha(t)} \left( \int_{t_0}^s b(s, \tau) d\tau \right) ds \right], \quad t_0 \leq t \leq b_1.$$

This completes the proof.  $\square$

REMARK 2.15. The method of the proof of Theorem 2.14 is similar to that of Lemma 2.1 [14].

We obtain the following result [5, Theorem 2.3] as a corollary of Theorem 2.14.

COROLLARY 2.16. *Let  $w_i, i = 1, 2$ , be continuous nondecreasing functions on  $\mathbb{R}^+$  and positive on  $(0, \infty)$  such that  $w = \frac{w_2}{w_1}$  is nondecreasing on  $(0, \infty)$ . Let  $u, \lambda_i \in C(\mathbb{R}^+), i = 1, 2, 3$ , and  $c$  a positive constant. If*

$$u(t) \leq c + \int_{t_0}^t \lambda_1(s) w_1(u(s)) ds + \int_{t_0}^t \lambda_2(s) \left( \int_{t_0}^s \lambda_3(\tau) w_2(u(\tau)) d\tau \right) ds,$$

for  $t \in I$ , then for  $0 \leq t_0 \leq t < b_1$ ,

$$u(t) \leq W_2^{-1} \left[ W_2(c_1) + \int_{t_0}^t \hat{\lambda}(s) ds \right],$$

where  $\hat{\lambda}(t) = \lambda_2(t) \int_{t_0}^t \lambda_3(s) ds, W_1(u) = \int_c^u \frac{dz}{w_1(z)}, u > 0, W_2(u) = \int_{u_0}^u \frac{dz}{w_2(z)}, u_0 > 0$  and  $b_1$  is the largest number such that  $b_1 \geq t_0$  and

$$\begin{aligned} |\lambda_1|_{b_1} &= \int_{t_0}^{b_1} \lambda_1(s) ds \leq \int_c^\infty \frac{dz}{w_1(z)}, \\ |\hat{\lambda}|_{b_1} &= \int_{t_0}^{b_1} \hat{\lambda}(s) ds \leq \int_{c_1}^\infty \frac{dz}{w_2(z)}, c_1 = W_1^{-1}(|\lambda_1|_{b_1}). \end{aligned}$$

*Proof.* In Theorem 2.14, if we put  $\alpha(t) = t, a(t) = \lambda_1(t)$  and  $b(t, s) = \lambda_2(t) \lambda_3(s)$ , then we obtain the result.  $\square$

REMARK 2.17. If  $\frac{w_1}{w_2}$  (instead of  $\frac{w_2}{w_1}$  in Theorem 2.14) is a nondecreasing function on  $(0, \infty)$ , then Theorem 2.14 can be applied by changing the roles of  $w_1$  and  $w_2$ , and  $a$  and  $b$ , respectively. In this case, the conclusion is

$$u(t) \leq W_1^{-1} [W_1(c_2) + A(t)], c_2 = W_2^{-1}(|A|_{b_1}).$$

We obtain the integral inequality with differential equation of Bernoulli-type as a corollary of Theorem 2.14.

COROLLARY 2.18. *Let the assumptions be the same as in the Theorem 2.14. If*

$$u(t) \leq c + \int_{\alpha(t_0)}^{\alpha(t)} \left[ a(s)u(s) + \int_{\alpha(t_0)}^s b(s, \sigma)u^p(\sigma)d\sigma \right] ds, \quad (2.21)$$

where  $w_1(u) = u$  and  $w_2(u) = u^p$  with a positive constant  $p$ , then for  $0 \leq t_0 \leq t \leq b_1$ ,

$$\begin{aligned} u(t) &\leq \begin{cases} W_1^{-1} [W_1(c_2) + A(t)], & 0 < p \leq 1 \\ W_2^{-1} [W_2(c_1) + A(t)], & 1 < p \end{cases} \\ &= \begin{cases} c_2 \exp(A(t)), & 0 < p \leq 1 \\ [c_1^{1-p} + (1-p)A(t)]^{\frac{1}{1-p}}, & 1 < p. \end{cases} \end{aligned} \quad (2.22)$$

*Proof.* Note that the functions of  $W_2(u)$  and  $W_2^{-1}(u)$  are given by

$$W_2(u) = \begin{cases} \ln \frac{u}{c}, & p = 1 \\ \frac{1}{1-p} [u^{1-p} - c^{1-p}], & 0 < p \neq 1, \end{cases}$$

and

$$W_2^{-1}(u) = \begin{cases} c \exp(u), & p = 1 \\ [(1-p)u + c^{1-p}]^{\frac{1}{1-p}}, & 0 < p \neq 1, \end{cases}$$

respectively. □

COROLLARY 2.19. [12, Theorem 2.1] *Let the assumptions be the same as in the Lemma 2.4. If*

$$u(t) \leq c + \int_{\alpha(t_0)}^{\alpha(t)} \left[ a(s)w(u(s)) + \int_{\alpha(t_0)}^s b(s, \sigma)w(u(\sigma))d\sigma \right] ds,$$

for  $t \in I$ , then for  $t_0 \leq t \leq b_1$ ,

$$u(t) \leq W^{-1} \left[ W(c) + \int_{\alpha(t_0)}^{\alpha(t)} \left( a(s) + \int_{\alpha(t_0)}^s b(s, \sigma) d\sigma \right) ds \right],$$

where  $W^{-1}$  is the inverse function of  $W$  defined by Lemma 2.4.

*Proof.* The proof can be accomplished by the same method as in Theorem 2.14. Here we omit the details.  $\square$

Consider the integrodifferential equation with the given initial condition

$$x'(t) = F(t, x(t-h(t)), \int_{t_0}^t f(t, \sigma, x(\sigma-h(\sigma))) d\sigma), x(t_0) = x_0, \quad (2.23)$$

where  $f \in C(I \times I \times \mathbb{R}, \mathbb{R})$ ,  $F \in C(I \times \mathbb{R}^2, \mathbb{R})$  and  $h \in C^1(I)$  is nondecreasing with  $t-h(t) \geq 0$ ,  $h'(t) < 1$ ,  $h(t_0) = 0$ . The following example give an estimate on the solution of (2.23).

EXAMPLE 2.20. Suppose that  $f$  and  $F$  satisfy

$$|f(t, s, x)| \leq b(t, s)w_2(|x|), \quad |F(t, x, z)| \leq a(t)w_1(|x|) + |z|, \quad (2.24)$$

where  $a(t), b(t, s), w_1, w_2$  are as defined in Theorem 2.14. Let  $M = \max_{t \in I} \frac{1}{1-h'(t)}$ . If  $x(t)$  is any solution of (2.23), then for  $t_0 \leq t \leq b_1$ ,

$$|x(t)| \leq W_2^{-1} \left[ W_2(c_1) + \int_{t_0}^{t-h(t)} \left[ \int_{t_0}^s b(s, \sigma) d\sigma \right] ds \right], \quad (2.25)$$

where  $W_1(u) = \int_c^u \frac{dz}{w_1(z)}$ ,  $u > 0$ ,  $c = |x_0|$ ,  $W_2(u) = \int_{u_0}^u \frac{dz}{w_2(z)}$ ,  $u_0 > 0$ , and  $b_1$  is the largest number such that  $b_1 \geq t_0$  and

$$\begin{aligned} |a|_{b_1} &= \int_{t_0}^{b_1} a(s) ds \leq \int_c^\infty \frac{dz}{w_1(z)}, \\ |A|_{b_1} &= \int_{t_0}^{b_1} A(s) ds \leq \int_{c_1}^\infty \frac{dz}{w_2(z)}, \quad c_1 = W_1^{-1}(|a|_{b_1}). \end{aligned}$$

*Proof.* Put  $t-h(t) = \alpha(t)$ . Then  $\alpha(t)$  is invertible. The solution  $x(t)$  of (2.23) can be written as

$$x(t) = x_0 + \int_{t_0}^t F \left( s, x(s-h(s)), \int_{t_0}^s f(s, \sigma, x(\sigma-h(\sigma))) d\sigma \right) ds.$$

Using the change of variables and the condition (2.24), we have

$$\begin{aligned} |x(t)| &\leq |x_0| + \left| \int_{\alpha(t_0)}^{\alpha(t)} F(\alpha^{-1}(s), x(s), \right. \\ &\quad \left. \int_{\alpha(t_0)}^{\alpha(\alpha^{-1}(s))} f(\alpha^{-1}(s), \alpha^{-1}(\sigma), x(\sigma)) \frac{d\sigma}{1 - h'(\alpha^{-1}(\sigma))} \right) \frac{ds}{1 - h'(\alpha^{-1}(s))} \Big| \\ &\leq |x_0| + \int_{\alpha(t_0)}^{\alpha(t)} \left[ \hat{a}(s) w_1(|x(s)|) + \int_{\alpha(t_0)}^s \hat{b}(s, \sigma) w_2(|x(\sigma)|) d\sigma \right] ds, \end{aligned}$$

where  $\hat{a}(t) = Ma(\alpha^{-1}(t))$  and  $\hat{b}(t, s) = M^2b(\alpha^{-1}(t), \alpha^{-1}(s))$ . Now, a suitable application of the inequality in Theorem 2.14 yields the required estimate in (2.25).  $\square$

REMARK 2.21. If we set  $w_1(u) = w_2(u) = u$  in Example 2.20, then we have

$$|x(t)| \leq |x_0| \exp \left( \int_{t_0}^{t-h(t)} (\hat{a}(s) + \int_{t_0}^s \hat{b}(s, \sigma) d\sigma) ds \right),$$

where  $\hat{a}(t) = Ma(\alpha^{-1}(t))$  and  $\hat{b}(t, s) = M^2b(\alpha^{-1}(t), \alpha^{-1}(s))$ ,  $\alpha(t) = t - h(t)$ . Also, we note that the inequality in Theorem 2.14 can be used to study the uniqueness and the continuous dependence of solutions of integrodifferential equation (2.23) on initial values. See Theorems 4.2 and 4.3 in [12] for details.

## References

- [1] R. P. Agarwal, On an integral inequality in  $n$  independent variables, *J. Math. Anal. Appl.* **85** (1982), 192–196.
- [2] R. P. Agarwal and E. Thandapani, Remarks on generalizations of Gronwall's inequality, *Chinese J. Math.* **9** (1981), 1–22.
- [3] D. Bainov and P. Simeonov, *Integral Inequalities and Applications*, Academic Publishers, Dordrecht, 1992.
- [4] R. Bellman, *Stability Theory of Differential Equations*, McGraw-Hill, New York, 1953.
- [5] S. K. Choi, Y. H. Goo and N. J. Koo, Lipschitz stability and exponential asymptotic stability for the nonlinear functional differential systems, *Dynam. Systems Appl.* **6** (1997), 397–410.
- [6] U. D. Dhongade and S. G. Deo, A nonlinear generalization of Bihari's inequality, *Proc. Amer. Math. Soc.* **54** (1976), 211–216.
- [7] O. Lipovan, A retarded Gronwall-like inequality and its applications, *J. Math. Anal. Appl.* **252** (2000), 389–401.

- [8] R. Medina and M. Pinto, On the asymptotic behavior of solutions of a class of second order nonlinear differential equations, *J. Math. Anal. Appl.* **135** (1988), 399-405.
- [9] J. A. Oguntuase, On an inequality of Gronwall, *J. Inequal. Pure and App. Math.* **2**(1) (2001), Art. 9, 1-6.
- [10] B. G. Pachpatte, On note on Gronwall-Bellman inequality, *J. Math. Anal. Appl.* **44** (1973), 758-762.
- [11] B. G. Pachpatte, On some generalization of Bellman's lemma, *J. Math. Anal. Appl.* **51** (1975), 141-150.
- [12] B. G. Pachpatte, On some retarded integral inequalities and applications, *J. Inequal. Pure and App. Math.* **3**(2) (2002), Art. 18, 1-7.
- [13] M. Pinto, Perturbations of asymptotically stable differential systems, *Analysis*, **4** (1984), 161-175.
- [14] M. Pinto, Integral inequalities of Bihari-type and applications, *Funkcial. Ekvac.* **33** (1990), 387-404.
- [15] W. Zhang, Generalized exponential dichotomies and invariant manifolds for differential equations, *Adv. Math. Chin.* **22** (1993), 1-45.

\*

Department of Mathematics  
 Chungnam National University  
 Daejeon 305-764, Republic of Korea  
*E-mail:* skchoi@math.cnu.ac.kr

\*\*

National Institute for Mathematical Sciences  
 385-16, Doryong-Dong, Yuseong-Gu  
 Daejeon 305-340, Republic of Korea  
*E-mail:* tychoi@nims.re.kr

\*\*\*

Daejeon Science High School  
 Daejeon 305-338, Republic of Korea  
*E-mail:* kimdj623@hanmail.net

\*\*\*\*

Department of Mathematics  
 Chungnam National University  
 Daejeon 305-764, Republic of Korea  
*E-mail:* njkoo@math.cnu.ac.kr