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# COMMON STATIONARY POINTS FOR CONTRACTIVE TYPE MULTIVALUED MAPPINGS

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Abstract. Several common stationary point theorems for two classes of contractive type multivalued mappings in a complete bounded metric space are given. The results presented in this paper generalize and extend some known results in literature.

## 1. Introduction

Let $(X, d)$ be a metric space and $B(X)$ denote the set of all nonempty bounded subsets of $X$.

For $A, B \in X$, define $\delta(A, B) = \sup\{d(a, b) : a \in A,\, b \in B\}$ and $\delta(A) = \delta(A, A)$. If $A$ consists of a single point $a$, we write $\delta(A, B) = \delta(a, B)$. If $B$ also consists of a single point $b$, we write $\delta(A, B) = \delta(a, b) = d(a, b)$.

Let $\mathbb{R}^+ = [0, +\infty)$, $\mathbb{N}$ and $\omega$ denote the sets of positive integers and nonnegative integers, respectively. Let $\Phi = \{\phi : \phi : \mathbb{R}^+ \to \mathbb{R}^+$ is an upper semicontinuous and nondecreasing function satisfying $\phi(t) < t$ for each $t > 0\}$.

Let $F$ and $G$ be multivalued mappings from $(X, d)$ into $B(X)$. A point $x \in X$ is called *a common stationary point* of $F$ and $G$ if $Fx = Gx = \{x\}$. For $A \subseteq X$, let $FA = \bigcup_{a \in A} Fa$ and $GFA = G(FA)$. The mappings $F$ and $G$ are said to *commute* if $FGx = GFx$ for all $x \in X$.

Define $C_F = \{T : T$ is a mapping of from $X$ into $B(X)$ which commutes with $F\}$. It follows that $C_F \supseteq \{F^n : n \in \omega\}$, where $F^0 x = \{x\}$ for all $x \in X$.

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In 1983, Fisher [4] established a common fixed point theorem for continuous and commuting mappings $F$ and $G$ from $(X,d)$ into $B(X)$ satisfying

$$\begin{aligned}&\delta(F^pG^px, F^pG^py)\\ &\leq c\max\{\delta(F^qG^rx, F^sG^ty), \delta(F^qG^rx, F^sG^tx),\\ &\qquad \delta(F^qG^ry, F^sG^ty) : 0 \leq q,r,s,t \leq p\}\end{aligned} \tag{1.1}$$

for all $x, y \in X$, where $0 \leq c < 1$ and $p$ is a fixed positive integer.

In 1994, Ohta and Nikaido [6] obtained the existence of fixed point for a continuous self mapping $f$ on $(X,d)$ satisfying

$$d(f^kx, f^ky) \leq c\delta(\{f^it : t \in \{x,y\}, i \in \omega\}) \tag{1.2}$$

for all $x, y \in X$, where $0 \leq c < 1$ and $k$ is a fixed positive integer.

In 2000, Liu and Kang [5] proved some common stationary point theorems for commuting mappings $F$ and $G$ from $(X,d)$ into $B(X)$ satisfying one of the following conditions:

$$\delta(F^pG^px, F^qG^qy) \leq \phi(\delta(\textstyle\bigcup_{D\in C_{FG}} D\{x,y\})) \tag{1.3}$$

for all $x, y \in X$, where $\phi \in \Phi$ and $p, q$ are fixed positive integers;

$$\delta(F^px, G^qy) \leq \phi(\delta(\textstyle\bigcup_{D\in C_F\cap C_G} D\{x,y\})) \tag{1.4}$$

for all $x, y \in X$, where $\phi \in \Phi$ and $p, q$ are fixed positive integers.

The purpose of the paper is to study the existence of common stationary points for the commuting multivalued mappings $F$ and $G$ from $(X,d)$ into $B(X)$ satisfying one of the conditions below:

$$\begin{aligned}&\delta(F^pG^qx, F^sG^ty)\\ &\leq \phi(\delta(\textstyle\bigcup_{D\in C_F\cap C_G} D(\bigcup\{F^mG^ru : u \in \{x,y\}, m, r \in \omega\})))\end{aligned} \tag{1.5}$$

for all $x, y \in X$, where $\phi \in \Phi$ and $p, q, s, t$ are fixed nonnegative integers satisfying $p + q \geq 1$ and $s + t \geq 1$;

$$\delta(F^px, G^qy) \leq \phi(\delta(\textstyle\bigcup_{D\in C_F}\bigcup_{m\in\omega} DF^mx, \bigcup_{E\in C_G}\bigcup_{r\in\omega} EG^ry)) \tag{1.6}$$

for all $x, y \in X$, where $\phi \in \Phi$ and $p, q$ are fixed nonnegative integers satisfying $p+q \geq 1$. Under certain conditions, we establish two common stationary point theorems for the contractive type multivalued mappings $F$ and $G$ satisfying (1.5) and (1.6), respectively. Our results extend and unify several results due to Fisher [1, 2, 4] and Ohta and Nikaido [6].

It is evident that the conditions (1.5) and (1.6) completely independent of each other.

Recall some concepts and result in [3, 7].

Definition 1.1. ([3]) Let $\{A_n\}_{n\in\mathbb{N}}$ be a sequence of sets in $B(X)$ and $A \in B(X)$. The sequence $\{A_n\}_{n\in\mathbb{N}}$ is said to *converge* to the set $A$ if (1) each point $a \in A$ is the limit of some convergent sequence $\{a_n\}_{n\in\mathbb{N}}$, where $a_n \in A_n$ for $n \in \mathbb{N}$;

(2) for arbitrary $\epsilon > 0$, there exists $k \in \mathbb{N}$ such that $A_n \subseteq A_\epsilon$ for $n > k$, where $A_\epsilon$ is the union of all open spheres with centers in $A$ and radius $\epsilon$.

Definition 1.2. ([3]) Let $F$ be a multivalued mapping of $(X, d)$ into $B(X)$. The mapping $F$ is said to be *continuous* in $X$ if whenever $\{x_n\}_{n\in\mathbb{N}}$ is a sequence of points in $X$ converging to $x \in X$, the sequence $\{Fx_n\}_{n\in\mathbb{N}}$ in $B(X)$ converges to $Fx \in B(X)$.

Lemma 1.1. ([3]) *If $\{A_n\}_{n\in\mathbb{N}}$ and $\{B_n\}_{n\in\mathbb{N}}$ are sequences of bounded subsets of a complete metric space $(X, d)$ which converge to the bounded subsets $A$ and $B$, resp., then the sequence $\{\delta(A_n, B_n)\}_{n\in\mathbb{N}}$ converges to $\delta(A, B)$.*

Lemma 1.2. ([7]) *Let $\phi \in \Phi$. Then $\phi(t) < t$ for each $t > 0$ if and only if $\lim_{n\to\infty} \phi^n(t) = 0$, where $\phi^n$ denotes the $n$-times composition of $\phi$.*

## 2. Common stationary points

We are now ready to prove our main results.

Theorem 2.1. *Let $(X, d)$ be a complete bounded metric space, $F$and $G$ be continuous and commuting mappings from $(X, d)$ into $B(X)$ satisfying*

(1.5). *Then $F$ and $G$ have a unique common stationary point $z \in X$ and the sequence $\{F^nG^nx\}_{n\in\omega}$ converges to $\{z\}$ for all $x \in X$.*

*Proof.* Let $A, B$ be in $B(X)$. By (1.5) we have

$$\begin{aligned}&\delta(F^pG^qa, F^sG^tb)\\ &\leq \phi(\delta(\textstyle\bigcup_{D\in C_F\cap C_G} D(\bigcup\{F^mG^ru : u \in \{a,b\}, m, r \in \omega\})))\end{aligned}$$

for all $a \in A$, $b \in B$, which implies that

$$\begin{aligned}&\delta(F^pG^qA, F^sG^tB)\\ &\leq \phi(\delta(\textstyle\bigcup_{D\in C_F\cap C_G} D(\bigcup\{F^mG^r(A\cup B) : m, r \in \omega\}))).\end{aligned} \tag{2.1}$$

Let $M = \delta(X)$, $k = p+q+s+t$ and $X_n = F^nG^nX$ for each $n \in \omega$. Choose $x_n \in X_n$ for each $n \in \omega$. Let $n$ be fixed in $\mathbb{N}$. It is clear that it can be written as

$$n = kj_n + i_n, \quad 0 \leq i_n < k,\ j_n \in \omega. \tag{2.2}$$

It follows from (2.1) and (2.2) that

$$\begin{aligned}&\delta(X_n)\\ &= \delta(F^nG^nX, F^nG^nX)\\ &= \delta(F^pG^q(F^{k+i_n-p}G^{k+i_n-q}X_{k(j_n-1)}),\\ &\qquad F^sG^t(F^{k+i_n-s}G^{k+i_n-t}X_{k(j_n-1)}))\\ &\leq \phi(\delta(\textstyle\bigcup_{D\in C_F\cap C_G} D(\bigcup\{F^mG^r(F^{k+i_n-p}G^{k+i_n-q}X_{k(j_n-1)}\\ &\qquad \textstyle\bigcup F^{k+i_n-s}G^{k+i_n-t}X_{k(j_n-1)}) : m, r \in \omega\})))\\ &= \phi(\delta(\textstyle\bigcup_{D\in C_F\cap C_G}\bigcup\{F^{k(j_n-1)}G^{k(j_n-1)}F^{k+i_n-p+m}G^{k+i_n-q+r}DX\\ &\qquad \textstyle\bigcup F^{k(j_n-1)}G^{k(j_n-1)}F^{k+i_n-s+m}G^{k+i_n-t+r}DX : m, r \in \omega\}))\\ &\leq \phi(\delta(X_{k(j_n-1)})),\end{aligned}$$

which together with $X_n \subseteq X_{n-1}$ yields that

$$\begin{aligned}\delta(X_n) &\leq \delta(X_{kj_n}) \leq \phi(\delta(X_{k(j_n-1)}))\\ &\leq \cdots \leq \phi^{j_n-1}(\delta(X_k)) \leq \phi^{j_n}(M).\end{aligned} \tag{2.3}$$

For any $m > n > k$, by (2.2) and (2.3) we have

$$d(x_n, x_m) \leq \delta(X_n, X_m) \leq \delta(X_n) \leq \phi^{j_n}(M). \tag{2.4}$$

It follows from (2.4) and $\phi \in \Phi$ that $\{x_n\}_{n\in\omega}$ is a Cauchy sequence. By completeness of $X$ we infer that there exists a point $z \in X$ such that $x_n \to z$ as $n \to \infty$.

It follows from (2.2) and (2.3) that

$$\begin{aligned}\delta(z, X_n) &\leq d(z, x_m) + \delta(x_m, X_n) \leq d(z, x_m) + \delta(X_n)\\ &\leq d(z, x_m) + \phi^{j_n}(M)\end{aligned}$$

for $m, n \in \mathbb{N}$ with $m > n$. Letting $m$ tend to infinity, we obtain that

$$\delta(z, X_n) \leq \phi^{j_n}(M), \quad \forall n \in \mathbb{N}. \tag{2.5}$$

Since $F$ is continuous and $x_n \to z$ as $n \to \infty$, it follows that $\{Fx_n\}_{n\in\omega}$ converges to $\{Fz\}$. Note that $Fx_n \subseteq F(F^nG^nX) = F^nG^nFX \subseteq X_n$ for all $n \in \mathbb{N}$. It follows that

$$\delta(z, Fx_n) \leq \delta(z, X_n), \quad \forall n \in \mathbb{N}. \tag{2.6}$$

Taking $n \to \infty$, in view of (2.5) and (2.6), we have

$$\delta(z, Fz) \leq \phi^{j_n}(M) \to 0,$$

that is, $Fz = \{z\}$. Similarly, we have $Gz = \{z\}$.

Suppose that $F$ and $G$ have a second common stationary point $w \in X$. Thus $\{u\} = F^nG^nu \subseteq X_n$ for $u \in \{z, w\}$ and $n \in \omega$. In view of (2.3), we infer that $d(z, w) \leq \delta(X_n) \leq \phi^{j_n}(M) \to 0 \quad$ as $n \to \infty$, which implies that $z = w$. That is, $F$ and $G$ have a unique common stationary point $z \in X$.

For $x \in X$ and $n \in \omega$, choose $y_n \in F^nG^nx$. It follows that

$$\begin{aligned}d(y_n, z) &\leq \delta(F^nG^nx, z) \leq \delta(X_n, z)\\ &\leq \delta(X_n) \leq \phi^{j_n}(M) \to 0 \quad \text{as } n \to \infty,\end{aligned}$$

which yields that $\{F^nG^nx\}_{n\in\omega}$ converges to $\{z\}$. This completes the proof. □

REMARK 2.1. Theorem 2.1 extends Theorem 4 in [1], Theorem 2 in [2], Theorem 1 in [4] and Theorem 3 in [6].

THEOREM 2.2. *Let* $(X, d)$ *be a complete bounded metric space,* $F$ *and* $G$ *be continuous mappings from* $(X, d)$ *into* $B(X)$ *satisfying* (1.6). *Then* $F$ *and* $G$ *have a unique common stationary point* $z \in X$ *and the sequences* $\{F^n x\}_{n\in\omega}$ *and* $\{G^n x\}_{n\in\omega}$ *converge to* $\{z\}$ *for all* $x \in X$.

*Proof.* Let $M = \delta(X)$, $k = p + q$, $X_n = F^n X$ and $Y_n = G^n X$ for every $n \in \omega$. Choose $x_n \in X_n$, $y_n \in Y_n$ for each $n \in \omega$. Let $n \in \mathbb{N}$ be fixed. Then, we note that (2.2) holds. In light of (1.6) and (2.2), we conclude that

$$\begin{aligned}
\delta(X_n, Y_n) &= \delta(F^n X, G^n X) \\
&= \delta(F^p(F^{q+i_n} X_{k(j_n-1)}), G^q(G^{p+i_n} Y_{k(j_n-1)})) \\
&\le \phi(\delta(\textstyle\bigcup_{D\in C_F}\bigcup_{m\in\omega} DF^m F^{q+i_n} X_{k(j_n-1)}, \\
&\qquad \textstyle\bigcup_{E\in C_G}\bigcup_{r\in\omega} EG^r G^{p+i_n} Y_{k(j_n-1)})) \\
&= \phi(\delta(\textstyle\bigcup_{D\in C_F}\bigcup_{m\in\omega} F^{k(j_n-1)} F^{m+q+i_n} DX, \\
&\qquad \textstyle\bigcup_{E\in C_G}\bigcup_{r\in\omega} G^{k(j_n-1)} G^{r+p+i_n} EY)) \\
&\le \phi(\delta(X_{k(j_n-1)}, Y_{k(j_n-1)})),
\end{aligned}$$

which implies that

$$\begin{aligned}
\delta(X_n, Y_n) &\le \delta(X_{kj_n}, Y_{kj_n}) \le \phi(\delta(X_{k(j_n-1)}, Y_{k(j_n-1)})) \\
&\le \cdots \le \phi^{j_n}(\delta(X)) = \phi^{j_n}(M).
\end{aligned} \tag{2.7}$$

For any $m > n > k$, by (2.2) and (2.7) we have

$$\begin{aligned}
d(x_n, x_m) &\le d(x_n, y_m) + d(y_m, x_m) \\
&\le \delta(X_n, Y_n) + \delta(Y_n, X_n) \\
&\le 2\phi^{j_n}(M) \to 0 \quad \text{as } n \to \infty,
\end{aligned}$$

which yields that $\{x_n\}_{n\in\omega}$ is a Cauchy sequence and hence $\lim_{n\to\infty} x_n = z$ for some $z \in X$ by completeness of $X$. Similarly, $\lim_{n\to\infty} y_n = w$ for some $w \in X$.

It follows from (2.2) and (2.7) that

$$\begin{aligned}\delta(z, X_n) &\le d(z, x_m) + \delta(x_m, X_n)\\ &\le d(z, x_m) + \delta(x_m, y_m) + \delta(y_m, X_n)\\ &\le d(z, x_m) + 2\phi^{j_n}(M)\end{aligned}$$

for $m, n \in \mathbb{N}$ with $m > n$. Letting $m$ tend to infinity, we obtain that

$$\delta(z, X_n) \le 2\phi^{j_n}(M), \quad \forall n \in \mathbb{N}. \tag{2.8}$$

As in the proof of Theorem 2.1, we conclude that $Fz = \{z\}$. Similarly, $Gw = \{w\}$. Furthermore, (2.2) and (2.7) ensure that

$$\begin{aligned}d(z, w) = \delta(Fz, Gw) &\le \delta(X_n, Y_n)\\ &\le \phi^{j_n}(M) \to 0 \quad \text{as } n \to \infty,\end{aligned}$$

which gives that $z = w$. Hence $Fz = \{z\} = Gz$.

Suppose that $F$ and $G$ have a second common stationary point $v$. Thus $\{u\} = F^n u \subseteq X_n$ and $\{u\} = G^n u \subseteq Y_n$ for $u \in \{z, v\}$ and $n \in \omega$. In view of (2.7), we infer that

$$d(z, v) \le \delta(X_n, Y_n) \le \phi^{j_n}(M) \to 0 \quad \text{as } n \to \infty,$$

which means that $z = v$. That is, $F$ and $G$ have a unique common stationary point $z$.

For $x \in X$ and $n \in \omega$, choose $a_n \in F^n x$. It follows from (2.2) and (2.7) that

$$\begin{aligned}d(a_n, z) &\le \delta(F^n x, z) \le \delta(X_n, z)\\ &\le \delta(X_n, Y_n) \le \phi^{j_n}(M) \to 0 \quad \text{as } n \to \infty,\end{aligned}$$

which implies that $\{F^n x\}_{n\in\omega}$ converges to $\{z\}$. Similarly, $\{G^n x\}_{n\in\omega}$ converges to $\{z\}$. This completes the proof. □

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