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TUBES OF WEINGARTEN TYPES IN A EUCLIDEAN 3-SPACE

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Abstract. In this paper, we study a tube in a Euclidean 3-space satisfying some equation in terms of the Gaussian curvature, the mean curvature and the second Gaussian curvature.

1. Introduction

Let f and g be smooth functions on a surface M in a Euclidean 3-space $\mathbb{E}^3$. The Jacobi function $\Phi(f,g)$ formed with f,g is defined by $\Phi(f,g) = \det\begin{pmatrix} f_s & f_t \\ g_s & g_t \end{pmatrix}$, where $f_s = \frac{\partial f}{\partial s}$ and $f_t = \frac{\partial f}{\partial t}$. In particular, a surface satisfying the Jacobi equation $\Phi(K,H) = 0$ with respect to the Gaussian curvature K and the mean curvature H on a surface M is called a *Weingarten surface* or a *W-surface*. Also, if a surface satisfies a linear equation with respect to K and H, that is, $aK + bH = c$ $(a,b,c \in \mathbb{R}, (a,b,c) \neq (0,0,0))$, then it is said to be a *linear Weingarten surface* or a *LW-surface*.
When the constant $b = 0$, a linear Weingarten surface M reduces to a surface with constant Gaussian curvature. When the constant $a = 0$, a linear Weingarten surface M reduces to a surface with constant mean curvature. In such a sense, the linear Weingarten surfaces can be regarded as a natural generalization of surfaces with constant Gaussian curvature or with constant mean curvature.

Several geometers ([3, 4, 6, 11, 12, 13]) have studied W-surfaces and LW-surfaces and obtained many interesting results.

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If the second fundamental form II of a surface M in $\mathbb{E}^3$ is non-degenerate, then it is regarded as a new (pseudo-) Riemannian metric. Therefore, the Gaussian curvature K_{II} of non-degenerate second fundamental form II can be defined formally on the Riemannian or pseudo-Riemannian manifold (M, II). We call the curvature K_{II} the *second Gaussian curvature* on M.

For a pair $(X, Y), X \neq Y$, of the curvatures K, H and K_{II} of M in $\mathbb{E}^3$, if M satisfies $\Phi(X, Y) = 0$ and $aX + bY = c$, then it said to be a (X, Y)-*Weingarten surface* and (X, Y)-*linear Weingarten surface*, respectively.
For study of these surfaces, W. Kühnel ([11]) and G. Stamou ([13]) investigate ruled (X, Y)-Weingarten surface in a Euclidean 3-space $\mathbb{E}^3$. Also, C. Baikoussis and Th. Koufogiorgos ([1]) studied helicoidal (H, K_{II})-Weingarten surfaces. F. Dillen and W. Kühnel ([3]) and F. Dillen and W. Sodsiri ([4, 5]) gave a classification of ruled (X, Y)-Weingarten surface in a Minkowski 3-space $\mathbb{E}^3_1$, where $X, Y \in \{K, H, K_{II}\}$. D. Koutroufiotis ([10]) and Th. Koufogiorgos and T. Hasanis ([9]) investigate closed ovaloid (X, Y)-linear Weingarten surface in $\mathbb{E}^3$. D. W. Yoon ([14]) and D. E. Blair and Th. Koufogiorgos ([2]) classified ruled (X, Y)-linear Weingarten surface in $\mathbb{E}^3$.

Following the Jacobi equation and the linear equation with respect to the Gaussian curvature K, the mean curvature H and the second Gaussian curvature K_{II}, an interesting geometric question is raised:

"*Classify all surfaces in a Euclidean 3-space and a Minkowski 3-space satisfying the conditions*

$$\Phi(X, Y) = 0, \tag{1.1}$$

$$aX + bY = c, \tag{1.2}$$

where $X, Y \in \{K, H, K_{II}\}$, $X \neq Y$ *and* $(a, b, c) \neq (0, 0, 0)$."

In this paper, we would like to contribute the solution of the above question, by studying this question for a tube in a Euclidean 3-space $\mathbb{E}^3$.

Throughout this paper, we assume that all objects are smooth and all surfaces are connected unless stated otherwise.

2. Preliminaries

We denote a surface M in $\mathbb{E}^3$ by

$$x(s, t) = (x_1(s, t), x_2(s, t), x_3(s, t)).$$

Let U be the standard unit normal vector field on a surface M defined by $U = \frac{x_s \times x_t}{||x_s \times x_t||}$, where $x_s = \frac{\partial x(s,t)}{\partial s}$. Then the first fundamental form I and the second fundamental form II of a surface M are defined by, respectively

$$I = Eds^2 + 2Fdsdt + Gdt^2,$$
$$II = eds^2 + 2fdsdt + gdt^2,$$

where

$$E = \langle x_s, x_s \rangle, \quad F = \langle x_s, x_t \rangle, \quad G = \langle x_t, x_t \rangle,$$
$$e = \langle x_{ss}, U \rangle, \quad f = \langle x_{st}, U \rangle, \quad g = \langle x_{tt}, U \rangle.$$

On the other hand, the Gaussian curvature K and the mean curvature H are given by, respectively

$$K = \frac{eg - f^2}{EG - F^2},$$
$$H = \frac{1}{2}\frac{Eg - 2Ff + Ge}{EG - F^2}.$$

From Brioschi's formula in a Euclidean 3-space, we are able to compute K_{II} of a surface by replacing the components of the first fundamental form E, F, G by the components of the second fundamental form e, f, g respectively in Brioschi's formula. Consequently, the second Gaussian curvature K_{II} of a surface is defined by (cf. [2].)

$$K_{II} = \frac{1}{(eg - f^2)^2} \left\{ \begin{vmatrix} -\frac{1}{2}e_{tt} + f_{st} - \frac{1}{2}g_{ss} & \frac{1}{2}e_s & f_s - \frac{1}{2}e_t \\ f_t - \frac{1}{2}g_s & e & f \\ \frac{1}{2}g_t & f & g \end{vmatrix} - \begin{vmatrix} 0 & \frac{1}{2}e_t & \frac{1}{2}g_s \\ \frac{1}{2}e_t & e & f \\ \frac{1}{2}g_s & f & g \end{vmatrix} \right\}. \tag{2.1}$$

REMARK 2.1. (cf. [1]) It is well known that a minimal surface has vanishing second Gaussian curvature but that a surface with vanishing second Gaussian curvature need not be minimal.

3. Tubes of Weingarten types

Let $\gamma : (a, b) \to \mathbb{E}^3$ be a smooth unit speed curve of finite length which is topologically imbedded in $\mathbb{E}^3$. The total space N_γ of the normal bundle of $\gamma((a, b))$ in $\mathbb{E}^3$ is naturally diffeomorphic to the direct product $(a, b) \times \mathbb{E}^2$ via the translation along γ with respect to the induced normal connection. For a sufficiently small $r > 0$, the tube of radius r about the curve γ is the set :

$$T_r(\gamma) = \{\exp_{\gamma(t)} v| \quad v \in N_{\gamma(t)}, ||v|| = r, a < t < b\}.$$

For a sufficiently small $r > 0$, the tube $T_r(\gamma)$ is a smooth surface in $\mathbb{E}^3$. Let $\mathbf{t}, \mathbf{n}, \mathbf{b}$ denote the Frenet frame of the unit speed curve $\gamma = \gamma(t)$. Then the position vector of $T_r(\gamma)$ can be expressed as

$$x = x(t, \theta) = \gamma(t) + r(\cos\theta\mathbf{n}(t) + \sin\theta\mathbf{b}(t)). \tag{3.1}$$

We denote by κ, τ the curvature and the torsion of the curve γ. Then Frent formula of $\gamma(t)$ is defined by

$$\begin{cases} \mathbf{t}' &= \kappa\mathbf{n}, \\ \mathbf{n}' &= -\kappa\mathbf{t} + \tau\mathbf{b}, \\ \mathbf{b}' &= -\tau\mathbf{n}, \end{cases} \tag{3.2}$$

where the prime denotes the differentiation with respect to t. Furthermore, we have the natural frame $\{x_s, x_t\}$ given by

$$\begin{aligned} x_t &= (1 - r\kappa\cos\theta)\mathbf{t} - r\tau\sin\theta\mathbf{n} + r\tau\cos\theta\mathbf{b} \\ &= \alpha\mathbf{t} + r\tau\mathbf{v}, \\ x_\theta &= -r\sin\theta\mathbf{n} + r\cos\theta\mathbf{b} = r\mathbf{v}, \end{aligned}$$

where we put $\alpha = 1 - r\kappa(t)\cos\theta$ and $\mathbf{v} = -\sin\theta\mathbf{n}(t) + \cos\theta\mathbf{b}(t)$. From which the components of the first fundamental form are

$$E = \alpha^2 + r^2\tau^2, \quad F = r^2\tau, \quad G = r^2. \tag{3.3}$$

On the other hand, the unit normal vector field U is obtained by

$$U = \frac{1}{||x_t \times x_\theta||}(x_t \times x_\theta) = -\cos\theta\mathbf{n} - \sin\theta\mathbf{b},$$

from this, the components of the second fundamental form of x are given by

$$e = r\tau^2 - \kappa\alpha\cos\theta, \quad f = r\tau, \quad g = r.$$

If the second fundamental form is non-degenerate, $eg - f^2 \neq 0$, that is, κ, α and $\cos\theta$ are nowhere vanishing. In this case, we can define formally the second Gaussian curvature K_{II} on $T_r(\gamma)$.

On the other hand, the Gauss curvature K, the mean curvature H and the second Gaussian curvature K_{II} are given by, respectively

$$K = -\frac{1}{r\alpha}\kappa\cos\theta, \tag{3.4}$$

$$H = \frac{1}{2r\alpha}(1 - 2r\kappa\cos\theta), \tag{3.5}$$

$$K_{II} = \frac{1}{4r\alpha^2\cos^2\theta}(4r^2\kappa^2\cos^4\theta - 6r\kappa\cos^3\theta + \cos^2\theta + 1). \tag{3.6}$$

Differentiating K, H and K_{II} with respect to t and θ, we get

$$K_t = -\frac{\kappa'\cos\theta}{r\alpha^2}, \quad K_\theta = \frac{\kappa\sin\theta}{r\alpha^2}, \tag{3.7}$$

$$H_t = -\frac{\kappa'\cos\theta}{2r\alpha^2}, \quad H_\theta = \frac{\kappa\sin\theta}{2\alpha^2}, \tag{3.8}$$

$$\begin{aligned}(K_{II})_t = \frac{1}{4r\alpha^4\cos\theta}(&-2r^3\kappa^2\kappa'\cos^4\theta + 6r^2\kappa\kappa'\cos^3\theta \\ &- 4r\kappa'\cos^2\theta - 2r^2\kappa\kappa'\cos\theta + 2r\kappa'), \\ (K_{II})_\theta = \frac{1}{4r\alpha^4\cos^4\theta}(&2r^3\kappa^3\cos^6\theta\sin\theta - 6r^2\kappa^2\cos^5\theta\sin\theta \\ &+ 4r\kappa\cos^4\theta\sin\theta + 4r^2\kappa^2\cos^3\theta\sin\theta \\ &- 6r\kappa\cos^2\theta\sin\theta + 2\cos\theta\sin\theta).\end{aligned} \tag{3.9}$$

Now, we investigate a tube $T_r(\gamma)$ in $\mathbb{E}^3$ satisfying the Jacobi equation $\Phi(X, Y) = 0$.

By using (3.7) and (3.8), $T_r(\gamma)$ satisfies identically the Jacobi equation $\Phi(K, H) = K_t H_\theta - K_\theta H_t = 0$. Therefore, $T_r(\gamma)$ is a Weingarten surface.

We consider a tube $T_r(\gamma)$ with non-degenerate second fundamental form in $\mathbb{E}^3$ satisfying the Jacobi equation

$$\Phi(K, K_{II}) = K_t(K_{II})_\theta - K_\theta(K_{II})_t = 0 \tag{3.10}$$

with respect to the Gaussian curvature K and the second Gaussian curvature K_{II}. Then, by (3.7) and (3.9) equation (3.10) becomes

$$r^2\kappa^2\kappa'\cos^2\theta + 4r\kappa\kappa'\cos\theta\sin\theta + \kappa'\sin\theta = 0.$$

Since this polynomial is equal to zero for every θ, all its coefficients must be zero. Therefore, we conclude that $\kappa' = 0$.

We suppose that a tube $T_r(\gamma)$ with non-degenerate second fundamental form in $\mathbb{E}^3$ is (H, K_{II})-Weingarten surface. Then it satisfies the equation

$$H_t(K_{II})_\theta - H_\theta(K_{II})_t = 0, \tag{3.11}$$

which implies

$$r^2\kappa^2\kappa'\cos^2\theta\sin\theta - 2r\kappa\kappa'\cos\theta\sin\theta + \kappa'\sin\theta = 0. \tag{3.12}$$

From (3.12) we can obtain $\kappa' = 0$.

Consequently, we have the following theorems:

THEOREM 3.1. *A tube in a Euclidean 3-space is a Weingarten surface.*

THEOREM 3.2. *Let $(X, Y) \in \{(K, K_{II}), (H, K_{II})\}$ and let $T_r(\gamma)$ be a tube defined by (3.1) in a Euclidean 3-space with non-degenerate second fundamental form. If $T_r(\gamma)$ is a (X, Y)-Weingarten surface, then the curvature of γ is a non-zero constant.*

THEOREM 3.3. *Under the same hypothesis and notation as in Theorem 3.2, if γ has non-zero constant torsion, then $T_r(\gamma)$ is generated by a circular helix γ.*

Finally, we study a tube $T_r(\gamma)$ in $\mathbb{E}^3$ satisfying a linear equation $aX + bY = c$.

First of all, we suppose that a tube $T_r(\gamma)$ in $\mathbb{E}^3$ is a linear Weingarten surface, that is, it satisfies the equation

$$aK + bH = c. \tag{3.13}$$

Then, by (3.4) and (3.5) we have

$$(2a\kappa + 2br\kappa - 2cr^2\kappa)\cos\theta - b + 2rc = 0.$$

Since $\cos\theta$ and 1 are linearly independent, we get

$$2a\kappa + 2br\kappa - 2cr^2\kappa = 0, \quad b = 2rc,$$

which imply

$$\kappa(a + cr^2) = 0.$$

If $a + cr^2 \neq 0$, then $\kappa = 0$. Thus, $T_r(\gamma)$ is an open part of a circular cylinder.

Next, suppose that a tube $T_r(\gamma)$ with non-degenerate second fundamental form in $\mathbb{E}^3$ satisfies the equation

$$aK + bK_{II} = c. \tag{3.14}$$

By (3.4) and (3.6), equation (3.14) becomes

$$(4ar\kappa^2 + 4br^2\kappa^2 - 4cr^3\kappa^2)\cos^4\theta - (4a\kappa + 6br\kappa - 8cr^2\kappa)\cos^3\theta$$
$$+ (b - 4cr)\cos^2\theta + b = 0.$$

Since the identity holds for every θ, all the coefficients must be zero. Therefore, we have

$$\begin{cases} 4ar\kappa^2 + 4br^2\kappa^2 - 4cr^3\kappa^2 = 0, \\ 4a\kappa + 6br\kappa - 8cr^2\kappa, \\ b - 4cr = 0, \\ b = 0. \end{cases}$$

Thus, we get $b = 0, c = 0$ and $\kappa = 0$. In this case, the second fundamental form of $T_r(\gamma)$ is degenerate.

Suppose that a tube $T_r(\gamma)$ with non-degenerate second fundamental form in $\mathbb{E}^3$ satisfies the equation

$$aH + bK_{II} = c. \tag{3.15}$$

By (3.5), (3.6) and (3.15), we have

$$(4ar^2\kappa^2 + 4br^2\kappa^2 - 4cr^3\kappa^2)\cos^4\theta - (6ar\kappa + 6br\kappa - 8cr^2\kappa)\cos^3\theta$$
$$+ (2a + b - 4cr)\cos^2\theta + b = 0,$$

from which we can obtain $b = 0$ and $\kappa = 0$.

Consequently, we have the following theorems:

THEOREM 3.4. *Let $T_r(\gamma)$ be a tube satisfying the linear equation $aK + bH = c$. If $a + br \neq 0$, then it is an open part of a circular cylinder.*

THEOREM 3.5. *Let $(X, Y) \in \{(K, K_{II}), (H, K_{II})\}$. Then there are no (X, Y)-linear Weingarten tubes in a Euclidean 3-space.*

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