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CHARACTERIZATIONS OF THE LOMAX, EXPONENTIAL AND PARETO DISTRIBUTIONS BY CONDITIONAL EXPECTATIONS OF RECORD VALUES

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Abstract. Let $\{X_n, n \geq 1\}$ be a sequence of independent and identically distributed random variables with absolutely continuous cumulative distribution function (cdf) $F(x)$ and probability density function (pdf) $f(x)$. Suppose $X_{U(m)}, m = 1, 2, \cdots$ be the upper record values of $\{X_n, n \geq 1\}$. It is shown that the linearity of the conditional expectation of $X_{U(n+2)}$ given $X_{U(n)}$ characterizes the lomax, exponential and pareto distributions.

1. Introduction

Let $\{X_n, n \geq 1\}$ be a sequence of independent and identically distributed random variables with absolutely continuous cumulative distribution function (cdf) $F(x)$ and probability density function (pdf) $f(x)$. Suppose $Y_n = max\{X_1, X_2, \cdots, X_n\}$ for $n \geq 1$. We say X_j is an upper record value of $\{X_n\}$ if $Y_j > Y_{j-1}$. By definition, X_1 is an upper record value. The indices at which the record values occur are given by the record value times $U(n)$ where $U(1) = 1$ and $U(n) = min\{k | k > U(n-1), X_k > X_U(n-1)\}$, $n > 1$.

We denoe by $X \in LOMAX(\mu, \sigma, v)$ if the random variable X has the corresponding cdf $F(x)$ of the form:

$$F(x) = \begin{cases} 1 - (1 + \frac{x-\mu}{\sigma})^{-v} & , x \geq \mu,\ \sigma > 0 \text{ and } v > 0 \\ 0 & , otherwise. \end{cases} \tag{1.1}$$

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Similarly, we denote by $X \in EXP(\lambda)$ the exponential distribution has the cdf $F(x)$ of the following form :

$$F(x) = \begin{cases} 1 - e^{-\lambda x} & , x > 0,\ \lambda > 0, \\ 0 & , otherwise. \end{cases} \tag{1.2}$$

For the Pareto distribution, $X \in PAR(\alpha, \beta)$, we take the following cdf :

$$F(x) = \begin{cases} 1 - (\frac{\alpha}{x})^{\beta} & , x \geq \alpha,\ \beta > 0, \\ 0 & , otherwise. \end{cases} \tag{1.3}$$

Using the conditional expectation of $X_{U(n+k)}$ given $X_{U(n)} = y$, we show that for the above three distributions

$$E(X_{U(n+k)} | X_{U(n)} = y) = ay + b \tag{1.4}$$

for some constants a and b.

Nagaraja(1977) characterized the Pareto distribution that if $E[h(X_{L_1} | X_{L_0} = y] = k(y)$ almost surely with respect to the distribution of X_{L_0} where $k(y)$ is a nondecreasing function on $[c, d]$, then $F(x)$ is uniquely determined. Lee(2002) showed that $X \in EXP(\lambda)$ if and only if $E[X_{U(n+i)} - X_{U(n)} | X_{U(m)} = y] = ic$, $i = 3, 4,\ n \geq m + 1$.

In this paper we show that relation (1.4) characterizes the lomax, exponential and pareto distributions for $k = 2$.

2. Results

THEOREM 2.1. *Let $\{X_n, n \geq 1\}$ be a sequence of i.i.d. random variables with common distribution function $F(x)$ which is absolutely continuous with pdf $f(x)$ and $E(X_n^2) < \infty$. If $E[X_{U(n+2)} \mid X_{U(n)} = y] = y$, then $F(x) = 1 - \frac{1}{\sqrt{2x+1}}$, $x > 0$.*

THEOREM 2.2. *Let $\{X_n, n \geq 1\}$ be a sequence of i.i.d. random variables with common distribution function $F(x)$ which is absolutely continuous with pdf $f(x)$ and $E(X_n^2) < \infty$. Then*

$$\begin{aligned} F(x) =& 1 - x^{-\alpha}\ , \lambda > 0,\ \alpha > 1 \ \textit{if and only if} \\ & E[X_{U(n+2)} \mid X_{U(n)} = y] = y + \frac{2}{\lambda}. \end{aligned} \tag{2.1}$$

THEOREM 2.3. *Let $\{X_n, n \geq 1\}$ be a sequence of i.i.d. random variables with common distribution function $F(x)$ which is absolutely continuous with pdf $f(x)$ and $E(X_n^2) < \infty$. Then*

$$\begin{aligned} F(x) =& 1 - x^{-\alpha},\ x > 1,\ \alpha > 1 \ \textit{if and only if} \\ &E[X_{U(n+2)} \mid X_{U(n)} = y] = (\frac{\alpha}{\alpha - 1})^2 y. \end{aligned} \tag{2.2}$$

3. Proofs

Proof of Theorem 2.1. Suppose $E[X_{U(n+2)} \mid X_{U(n)} = y] = y$. Using Ahsanullah formula(1995), we get the following equation

$$\frac{1}{1-F(y)} \int_y^\infty \left(ln \frac{1-F(y)}{1-F(x)} \right) x f(x)\, dx = y. \tag{3.1}$$

Since $F(x)$ is absolutely continuous, we can differentiate both sides of (3.1) with respect to y and simplify and we obtain the following equation

$$3 + \frac{(1-F(y))f'(y)}{f^2(y)} = 0, \qquad \text{i.e.} \ -3\frac{f(y)}{1-F(y)} = \frac{f'(y)}{f(y)}. \tag{3.2}$$

Integrating (3.2) with respect to y and using the boundary conditions $F(0) = 0$ and $f(0) = 1$, we get

$$(1 - F(y))^3 = f(y). \tag{3.3}$$

By the existence and uniqueness theorem of the differential equation with the prescribed initial conditions, we obtain $F(x) = 1 - \frac{1}{\sqrt{2x+1}}$ from (3.3).

This completes the proof. □

Proof of Theorem 2.2. If $F(x) = 1 - e^{-\lambda x}$, $\lambda > 0$, $x > 0$, then

$$\begin{aligned} E[X_{U(n+2)} \mid X_{U(n)} = y] &= e^{\lambda y} \int_y^\infty (ln \frac{e^{\lambda x}}{e^{\lambda y}}) x (\lambda e^{\lambda x}) dx \\ &= y + \frac{2}{\lambda}. \end{aligned} \tag{3.4}$$

Hence (2.1) holds. Conversely, suppose (2.1) holds. From Ahsanullah formula (1995), we can obtain the following equation

$$\frac{1}{1-F(y)} \int_y^\infty (ln \frac{1-F(y)}{1-F(x)}) x f(x) dx = y + \frac{2}{\lambda},\ for\ \lambda > 0. \tag{3.5}$$

Since $F(x)$ is absolutely continuous, we can differentiate both sides of (3.5) with respect to y and simplify and we get the following equation

$$3(1-F(y))f^2(y)-\frac{2}{\lambda}f^3(y)+(1-F(y))^2f'(y)=0. \tag{3.6}$$

Let $y=F(y)$ (i.e. $y'=f(y)$, $y''=f'(y)$). Then (3.6) expressed by the following form

$$3(1-y){y'}^2-\frac{2}{\lambda}(y')^3+(1-y)^2y''=0. \tag{3.7}$$

Therefore there exists a unique solution of the differential equation (3.7) that satisfies the initial conditions $y(0)=0$, $y'(0)=\lambda$ and $y''(0)=-\lambda^2$. By the existence and uniqueness theorem, we get $F(x)=1-e^{-\lambda x}$.

This completes the proof. □

Proof of Theorem 2.3. If $F(x)=1-e^{-\alpha x}$, $x>1$, $\alpha>1$, then

$$\begin{aligned} E[X_{U(n+2)}\mid X_{U(n)}=y] &= y^{\alpha}\int_y^{\infty}(ln\frac{x^{\alpha}}{y^{\alpha}})x(\alpha x^{-\alpha-1})dx \\ &= (\frac{\alpha}{\alpha-1})^2y. \end{aligned} \tag{3.8}$$

Hence (2.2) holds. Conversely, suppose (2.2) holds. From Ahsanullah formula (1995), we have

$$\frac{1}{1-F(y)}\int_y^{\infty}(ln\frac{1-F(y)}{1-F(x)})xf(x)dx=(\frac{\alpha}{\alpha-1})^2y,\ \ for\ \alpha>1. \tag{3.9}$$

Since $F(x)$ is absolutely continuous, we can differentiate both sides of (3.9) with respect to y and simplify and we get

$$\begin{aligned} &3(\frac{\alpha}{\alpha-1})^2(1-F(y))+\frac{1}{1-\alpha}yf(y) \\ &\qquad +(\frac{\alpha}{\alpha-1})^2\frac{(1-F(y))^2f'(y)}{f^2(y)}=0. \end{aligned} \tag{3.10}$$

Therefore, by the existence and uniqueness theorem, there exists a unique solution of the differential equation (3.10) that satisfies the initial conditions $F(1)=0$, $f(1)=\alpha$ and $f'(1)=-\alpha(\alpha+1)$. Thus we get $F(x)=1-x^{-\alpha}$ from (3.10).

This completes the proof. □

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