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ON A GENERALIZED UPPER BOUND FOR THE EXPONENTIAL FUNCTION

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Abstract. With the introduction of a new parameter $n \geq 1$, Kim generalized an upper bound for the exponential function that implies the inequality between the arithmetic and geometric means. By a change of variable, this generalization is equivalent to $\exp\left(\frac{n(x-1)}{n+x-1}\right) \leq \frac{n-1+x^n}{n}$ for real $n \geq 1$ and $x > 0$. In this paper, we show that this inequality is true for real $x > 1-n$ provided that n is an even integer.

1. Introduction and statement of result

In §4.2 of the classical treatise [1], the inequality between the arithmetic and geometric means is deduced from

$$1 + x \leq e^x.$$

This is the proof of "Pólya's dream" [5]. With a change of variable this can be rewritten as

$$e^x \leq \frac{1}{1-x} \tag{1.1}$$

for $x < 1$. Kim [3] established the following generalization of which (1.1) is the case $n = 1$. For convenience, we let

$$U(n,x) = 1 - \frac{1}{n} + \frac{1}{n}\left(\frac{1+\left(1-\frac{1}{n}\right)x}{1-\frac{x}{n}}\right)^n.$$

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Theorem 1.1. *For real $n \geq 1$ and*

$$-\frac{n}{n-1} < x < n,$$

we have

$$e^x \leq U(n, x)$$

with equality if and only if $x = 0$. Moreover, for $0 \leq x < 1$ and $1 \leq n \leq 2$ we have

$$e^x \leq U(n, x) \leq \frac{1}{1-x},$$

and for $x < 0$ and $0 < n \leq 1$ we have

$$e^x \leq \frac{1}{1-x} \leq U(n, x).$$

Here $U(n, x)$ is smoothly and densely algebraic in n, and improves previous tight bounds [2] and [4]. For the details, see [3]. The change of variable given by replacing x with

$$\frac{n(x-1)}{n+x-1}$$

plays an important role here. In fact, it is immediate that the first part of Theorem 1.1 is equivalent to

Theorem 1.2. *For real $n \geq 1$ and $x > 0$ we have*

$$\exp\left(\frac{n(x-1)}{n+x-1}\right) \leq \frac{n-1+x^n}{n} \tag{1.2}$$

with equality if and only if $x = 1$.

Our purpose in this paper is to study Theorem 1.2 further. We obtain Theorem 1.3 below that (1.2) is true for real $x > 1 - n$ provided that n is a positive even integer. Observe that the left of (1.2) is not defined at $x = 1 - n$.

Theorem 1.3. *For positive even integer n and real $x > 1 - n$, we have (1.2) with equality if and only if $x = 1$.*

2. Proof of Theorem 1.3

Let n be a positive even integer and $x \in (1 - n, \infty)$. For the proof of Theorem 1.3, we observe that (1.2) is equivalent to

$$g(x) := \frac{n(x-1)}{n+x-1} \leq \log\left(\frac{n-1+x^n}{n}\right) =: f(x). \tag{2.1}$$

Since both sides of (2.1) are zero when $x = 1$, we may apply the following lemma (proof omitted).

Lemma 2.1. *Let $f(x)$ and $g(x)$ be differentiable functions on a finite or infinite interval I containing 1 such that $f(1) = g(1)$, $g'(x) \geq f'(x)$ for $x < 1$ and $g'(x) \leq f'(x)$ for $x > 1$. Then $g(x) \leq f(x)$ on I.*

Now $f(x)$ and $g(x)$ are differentiable functions on an infinite interval $(1-n, \infty)$, and

$$g'(x) = \frac{n^2}{(n-1+x)^2} \qquad \text{and} \qquad f'(x) = \frac{nx^{n-1}}{n-1+x^n}.$$

To verify the hypothesis of Lemma 2.1, we need to show that $H(x)$ has the same sign as $(x-1)$, where

$$H(x) := f'(x) - g'(x) = \frac{nh(x)}{(x+n-1)^2(x^n+n-1)}$$

and

$$h(x) = x^{n+1} + (n-2)x^n + (n-1)^2x^{n-1} + n(1-n).$$

Note that $H(x)$ has the same sign as $h(x)$. So it is enough to show that $h(x)$ has the same sign as $(x-1)$. This follows from

$$\begin{aligned} h'(x) &= x^{n-2}\left[(n+1)x^2 + n(n-2)x + (n-1)^3\right] \\ &= x^{n-2}(n+1)\left[\left(x + \frac{n(n-2)}{2(n+1)}\right)^2 + \frac{(n^2-2)(3n^2-4n+2)}{4(n+1)^2}\right] \\ &> 0 \end{aligned}$$

for all $x \in (1-n, \infty)$ and $h(1) = 0$. $\square$

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