JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **22**, No. 1, March 2009

# A CHARACTERIZATION OF GAMMA DISTRIBUTION BY INDEPENDENT PROPERTY

Min-Young Lee* and Eun-Hyuk Lim**

Abstract. Let $\{X_n, n \geq 1\}$ be a sequence of independent identically distributed(i.i.d.) sequence of positive random variables with common absolutely continuous distribution function(cdf) $F(x)$ and probability density function(pdf) $f(x)$ and $E(X^2) < \infty$. The random variables $\frac{X_i \cdot X_j}{(\Sigma_{k=1}^n X_k)^2}$ and $\Sigma_{k=1}^n X_k$ are independent for $1 \leq i < j \leq n$ if and only if $\{X_n, n \geq 1\}$ have gamma distribution.

## 1. Introduction

The random variable $X$ is said to have a Gamma distribution with shape parameter $\alpha$ and scale parameter $\beta$ if

$$f(x; \alpha, \beta) = \begin{cases} \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta} & \text{for } x > 0, \\ 0 & \text{for } x \leq 0, \end{cases}$$

where $\alpha > 0$ and $\beta > 0$. The characteristic function of gamma distribution is given by

$$\phi\ (t; \alpha, \beta) = (1 - it\beta)^{-\alpha}.$$

Here $\alpha$ and $\beta > 0$ are two parameters.

Let $X$ and $Y$ be two independent non-degenerate positive random variables. Then Lukacs(1955) proved that $X/Y$ and $X + Y$ are independent if and only if $X$ and $Y$ are gamma distribution with the same scale parameter.

Using the moment, Findeisen(1978) characterized the gamma distribution. Also, Hwang and Hu(1999) proved a characterization of

Received September 25, 2008; Accepted February 13, 2009.

2000 Mathematics Subject Classification: Primary 60E05, 60E10.

Key words and phrases: independent identically distributed, a statistic scale-invariant, gamma distribution.

Correspondence should be addressed to Min-Young Lee, leemy@dankook.ac.kr

The present research was conducted by the research fund of Dankkok University in 2008.

the gamma distribution by the independence of the sample mean and the sample coefficient of variation. Recently, Lee and Lim(2007) presented characterizations of gamma distribution that the random variables $\Sigma_{k=1}^{n} X_k$ and $\dfrac{\Sigma_{k=1}^{m} X_k}{\Sigma_{k=1}^{n} X_k}$ are independent for $1 \le m < n$ if and only if $X_1, \cdots, X_n$ have gamma distribution.

In this paper, we obtain the characterization of gamma distribution by independent property of product and sum of random variables.

## 2. Main result

THEOREM 2.1. *Let $\{X_n, n \ge 1\}$ be a sequence of i.i.d. sequence of positive random variables with common absolutely cdf $F(x)$ and pdf $f(x)$ and $E(X^2) < \infty$. The random variables $\dfrac{X_i \cdot X_j}{(\Sigma_{k=1}^{n} X_k)^2}$ and $\Sigma_{k=1}^{n} X_k$ are independent for $1 \le i < j \le n$ if and only if $\{X_n, n \ge 1\}$ have gamma distribution.*

*Proof.* Since $\dfrac{X_i \cdot X_j}{(\Sigma_{k=1}^{n} X_k)^2}$ is a statistic scale-invariant, $\dfrac{X_i \cdot X_j}{(\Sigma_{k=1}^{n} X_k)^2}$ and $\Sigma_{k=1}^{n} X_k$ are independent for gamma variable [see Lukacs and Laha(1963)]. We have to prove the converse.

We denote the characteristic functions of $\dfrac{X_i \cdot X_j}{(\Sigma_{k=1}^{n} X_k)^2}$, $\Sigma_{k=1}^{n} X_k$ and $\left(\dfrac{X_i \cdot X_j}{(\Sigma_{k=1}^{n} X_k)^2}, \Sigma_{k=1}^{n} X_k\right)$ by $\phi_1(t)$,$\phi_2(s)$ and $\phi(t,s)$, respectively. The independence of $\dfrac{X_i \cdot X_j}{(\Sigma_{k=1}^{n} X_k)^2}$ and $\Sigma_{k=1}^{n} X_k$ is equivalent to

$$\phi(t,s) = \phi_1(t) \cdot \phi_2(s). \tag{1}$$

The left hand side of (1) becomes

$$\phi(t,s) = \int_0^\infty \cdots \int_0^\infty \exp\left\{ \frac{is(x_i \cdot x_j)}{(\Sigma_{k=1}^{n} x_k)^2} + it(\Sigma_{k=1}^{n} x_k) \right\} dF$$

where $dF = f(x_1)\cdots f(x_n) \quad dx_1 \ \cdots \ dx_n$. Also the right hand side of (1) becomes

$$\phi_1(t)\cdot\phi_2(s)=\int_0^\infty\cdots\int_0^\infty \exp\left\{\frac{is(x_i\cdot x_j)}{(\Sigma_{k=1}^n x_k)^2}\right\}dF \cdot\int_0^\infty\cdots\int_0^\infty \exp\{it(\Sigma_{k=1}^n x_k)\}dF.$$

Then (1) gives

$$\begin{aligned}&\int_0^\infty\cdots\int_0^\infty \exp\left\{\frac{is(x_i\cdot x_j)}{(\Sigma_{k=1}^n x_k)^2}+it(\Sigma_{k=1}^n x_k)\right\}dF\\ &=\int_0^\infty\cdots\int_0^\infty \exp\left\{\frac{is(x_i\cdot x_j)}{(\Sigma_{k=1}^n x_k)^2}\right\}dF \cdot\int_0^\infty\cdots\int_0^\infty \exp\{it(\Sigma_{k=1}^n x_k)\}dF.\end{aligned}\tag{2}$$

The integrals in (2) exist not only for reals $t$ and $s$ but also for complex values $t=u+iv, s=u^*+iv^*$, where $u$ and $u^*$ are reals, for which $v=Im(t)\geq 0$, $v^*=Im(s)\geq 0$ and they are analytic for all $t$, $s$ for $v=Im(t)>0$, $v^*=Im(s)>0$ [see Lukacs(1955)].

Differentiating (2) one time with respect to $s$ and then two times respect to $t$ and setting $s=0$, we get

$$\begin{aligned}&\int_0^\infty\cdots\int_0^\infty x_i\,x_j\exp\{it(\Sigma_{k=1}^n x_k)\}dF\\ &=\theta\int_0^\infty\cdots\int_0^\infty (\Sigma_{k=1}^n x_k)^2\exp\{it(\Sigma_{k=1}^n x_k)\}dF\end{aligned}\tag{3}$$

where $\theta=E\left[\frac{X_i\cdot X_j}{(\Sigma_{k=1}^n X_k)^2}\right]$ for $1\leq i<j\leq n$.

The random variable $\theta$ is bounded. Therefore all its moments exist.

Note that

$$\theta=E\left[\frac{X_1\cdot X_2}{(\Sigma_{k=1}^n X_k)^2}\right]=E\left[\frac{X_1\cdot X_3}{(\Sigma_{k=1}^n X_k)^2}\right]=\cdots=E\left[\frac{X_{n-1}\cdot X_n}{(\Sigma_{k=1}^n X_k)^2}\right]$$

for i.i.d. random variables $X_1,\cdots,X_n$.

Then we get the following equation by adding of all $\theta$ and multiplying 2

$$
\begin{aligned}
2 \cdot_n C_2 \cdot \theta &= E\left[\frac{2\Sigma\Sigma_{1\leq i<j\leq n} X_i X_j}{(\Sigma_{k=1}^n X_k)^2}\right] \\
&= E\left[\frac{1}{1+\dfrac{\Sigma_{k=1}^n X_k^2}{2\Sigma\Sigma_{1\leq i<j\leq n} X_i X_j}}\right].
\end{aligned} \tag{4}
$$

Note that, for $x_1 > 0, \cdots, x_n > 0$, $0 < 2\Sigma\Sigma_{1\leq i<j\leq n} x_i x_j \leq (n-1)(\Sigma_{k=1}^n x_k^2)$ and the equality on the right hand side occurs only if $x_n = \cdots = x_n$. By the assumed continuity of $F(x)$, $P(X_1 = \cdots = X_n) = 0$, so $\dfrac{\Sigma_{k=1}^n x_k^2}{2\Sigma\Sigma_{1\leq i<j\leq n} x_i x_j} > \dfrac{1}{n-1}$, that is, by (4), $0 < \theta < \dfrac{1}{n^2}$.

Let $\varphi(t)$ be the characteristic function of $F(x)$. Then

$$\varphi'(t) = i\int_0^\infty x \exp\{itx\} dF(x)$$

and

$$\varphi''(t) = -\int_0^\infty x^2 \exp\{itx\} dF(x).$$

We can express (3) as a differential equation for the characteristic function $\varphi(t)$ and get

$$(\varphi(t)')^2 \varphi(t)^{n-2} = \theta\{n\varphi''(t)(\varphi(t))^{n-1} + 2 \cdot_n C_2 (\varphi'(t))^2 \varphi(t)^{n-2}\}.$$

That is,

$$\frac{\varphi''(t)}{\varphi'(t)} = \frac{1-2\theta \cdot_n C_2}{n\theta}\frac{\varphi'(t)}{\varphi(t)}, \qquad 0 < \theta < \frac{1}{n^2}.$$

After integrating with the initial conditions $\varphi(0) = 1$, $\varphi'(0) = iE(X)$, we get

$$\varphi'(t) = iE(X)(\varphi(t))^{\frac{1-2\theta\cdot_n C_2}{n\theta}}, \quad \frac{1-2\theta \cdot_n C_2}{n\theta} > 1. \tag{5}$$

The solution of this differential equation (5) with the above initial conditions is

$$\varphi(t) = \left(1 - \frac{iE(X)}{\alpha}t\right)^{-\alpha}, \qquad \alpha = \frac{n\theta}{1-n^2\theta} > 0.$$

Therefore $F(x)$ is a gamma distribution.

$\square$

## References


[1] P. Findeisen, *A simple proof of a classical theorem which characterizes the gamma distribution,* Ann. Stat. **6** (1978), no. 5, 1165-1167

[2] T.Y. Hwang and C.Y. Hu, *On a characterizations of the gamma distribution: The independence of the sample mean and the sample coefficient of variation,* Ann. Inst. Stat. Math. **51** (1999), no. 4, 749-753.

[3] M.Y. Lee and E.H. Lim, *Characterization of gamma distribution*, J. Chungcheong Math. soc. **20** (2007), 411-418.

[4] E. Lukacs, *A Characterization of the gamma distribution,* Ann. Math. Stat. **17** (1955), 319-324.



*

Department of Mathematics
Dankook University
Cheonan 330-714, Republic of Korea
*E-mail*: `leemy@dankook.ac.kr`

**

Department of Mathematics
Dankook University
Cheonan 330-714, Republic of Korea
*E-mail*: `ehlim@dankook.ac.kr`