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SOME RESULTS ON STRONG π–REGULARITY

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Abstract. We will introduce some properties of strongly reduced near-rings and the notion of left π-regular near-ring. Also, we will study for right π-regular, strongly left π-regular, strongly right π-regular and strongly π-regular. Next, we may characterize the strongly π-regular near-rings with related strong reducibility.

1. Introduction

A near-ring R is an algebraic system $(R, +, \cdot)$ with two binary operations $+$ and $\cdot$ such that $(R, +)$ is a group (not necessarily abelian) with neutral element 0, $(R, \cdot)$ is a semigroup and $(a+b)c = ac+bc$ for all a, b, c in R. If R has a unity 1, then R is called *unital*. A near-ring R with the extra axiom $a0 = 0$ for all $a \in R$ is said to be *zero symmetric.*

We will use the following notations: Given a near-ring R, $R_0 = \{a \in R \mid a0 = 0\}$ which is called the *zero symmetric part* of R, $R_c = \{a \in R \mid a0 = a\}$ which is called the *constant part* of R. Obviously, we see that R_0 and R_c are subnear-rings of R, but R_d is a semigroup under multiplication.

Mason [3] introduced the notions of left and right regularities and characterized left regular zero-symmetric unital near-rings. Also, several authors ([2], [3], [4], [6] etc.) studied them. In particular, Reddy and Murty [6] observed that every left regular near-ring has some interesting property (*) with conditions (i) and (ii).

A near-ring R is called *(Von Neumann) regular* if for any element $a \in R$, there exists an element x in R such that $a = axa$. Such an element a is called *regular* [5].

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A near-ring R is said to be *left regular* if, for each $a \in R$, there exists $x \in R$ such that $a = xa^2$. Such an element a is called *left regular*. This concept is equivalent to the concept of strong regularity in [2]. Right regularity is defined in a symmetric way.

For more notations and basic results, we shall refer to Pilz [5].

2. Results

Also, we say that R is *reduced* if R has no nonzero nilpotent elements, that is, for each a in R, $a^n = 0$, for some positive integer n implies $a = 0$. McCoy proved that R is reduced if and only if for each a in R, $a^2 = 0$ implies $a = 0$.

A near-ring R is said to be *strongly reduced* if, for any $a \in R$, $a^2 \in R_c$ implies $a \in R_c$ which is defined in [1]. Obviously R is strongly reduced if and only if, for $a \in R$ and any positive integer n, $a^n \in R_c$ implies $a \in R_c$. We see that a strongly reduced near-ring is reduced, but not conversely. Clearly, if R is a zero-symmetric near-ring, then R is strongly reduced if and only if R is reduced [1].

We begin with to introduce the following basic properties of strong reducibility.

Lemma 1 [1]. *(1) Every strongly regular near-ring is strongly reduced.*

(2) Every right regular near-ring is strongly reduced.

(3) Every commutative integral near-ring is strongly reduced.

Lemma 2 [1]. *Let R be a strongly reduced near-ring. Then we have the following conditions.*

(1) If for any $a, b \in R$ with $ab \in R_c$, then $ba \in R_c$, and $\forall x \in R$, axb, $bxa \in R_c$. Furthermore, $ab^n \in R_c$ implies $ab \in R_c$, for each positive integer n.

(2) If for any $a, b \in R$ with $ab = 0$, then $ba = b0 = (ba)^2$. Moreover, $ab^n = 0$ implies $ab = 0$, for any positive integer n.

Lemma 3. *Let R be a strongly reduced near-ring. If for any $a, b \in R$ with $ab = 0$ and $a^2 = a0$, then $a = 0$.*

Proof. Suppose that for any $a, b \in R$ with $ab = 0$ and $a^2 = a0$. Then $a^2 = a0 \in R_c$. Strong reducibility implies that $a \in R_c$. Hence we obtain that $a = a0 = a0b = ab = 0$. □ □

From this Lemma 3, we have the following important statement.

COROLLARY 4. *Every strongly reduced near-ring is reduced.*

By Reddy and Murty [6], we say that a near-ring R has the property (*) if it satisfies the conditions:

(i) for any $a, b \in R$, $ab = 0$ implies $ba = b0$.

(ii) for $a \in R$, $a^3 = a^2$ implies $a^2 = a$.

Here, clearly we see that strong reducibility is equivalent to the condition (ii) and strong reducibility implies condition (i) by Lemma 2 (2).

A near-ring R is said to be *π-regular* if for each element $a \in R$, there exists a positive integer n such that a^n is a regular element, that is, $a^n = a^n x a^n$, for some $x \in R$. Such an element a is called *π-regular*.

A near-ring R is said to be *left π-regular* if, for each $a \in R$, there exists a positive integer n such that a^n is left regular. Such an element a is called *left π-regular*. Right π-regularity is defined in a symmetric way.

A near-ring R is called *strongly left π-regular* if R is left π-regular and π-regular, similarly, we can define strongly right π-regular. A strongly left π-regular and strongly right π-regular near-ring is called *strongly π-regular near-ring.*

Every regular near-ring is π-regular, but not conversely as following examples.

EXAMPLES 5. *Let $R = \{0, a, b, c\}$ be an additive Klein 4-group. This is a near-ring with the following multiplication table (p. 408 [5]):*

$\cdot$	0	a	b	c
0	0	0	0	0
a	0	0	a	a
b	0	a	c	b
c	0	a	b	c

This near-ring R is a zero-symmetric near-ring with identity c. Moreover, R is π-regular, but not regular. Indeed, $0 = 0a0$, $a^2 = a^2ba^2$, $b^4 = b^4ab^4$, $c^2 = c^2cc^2$, but a is not a regular element.

On the other hand, we can find lots of examples of left π-regular near-rings which are not left regular, also, for right π-regular, strongly right π-regular and strongly π-regular.

The purpose of this paper is to prove that the notions of left π-regularity, strongly left π-regularity, strongly right π-regularity and strong π-regularity are all equivalent conditions under strong reducibility, also to find some characterizations of strong regularity..

PROPOSITION 6. *Let R be a strongly reduced left π-regular near-ring. Then R is π-regular. Furthermore, R is strongly left π-regular.*

Proof. Let $a \in R$. Left π-regularity of R implies that $a^n = xa^{2n}$ for some $x \in R$ and some positive integer n. From this equation, we have that $(a^n - a^nxa^n)a^n = 0$. By Lemma 2 (2), $a^n(a^n - a^nxa^n) = a^n0$ and $a^nxa^n(a^n - a^nxa^n) = a^nxa^n0$. Thus we have

$$(a^n - a^nxa^n)^2 = a^n(a^n - a^nxa^n) - a^nxa^n(a^n - a^nxa^n) = (a^n - a^nxa^n)0.$$

This equality implies that $a^n - a^nxa^n = 0$ using Lemma 3. Consequently R is π-regular. $\square$ $\square$

PROPOSITION 7. *Let R be a strongly reduced left π-regular near-ring. Then R is right π-regular. Furthermore, R is strongly right π-regular.*

Proof. Let $a \in R$. Proposition 6 and left π-regularity of R imply that $a^n = xa^{2n} = a^nxa^n$ for some $x \in R$ and some positive integer n. From this last equation, we have that $(xa^n - a^nx)a^n = 0$ and $(xa^n - a^nx)a^nx = 0$. By Lemma 2 (2), we see that $a^n(xa^n - a^nx) = a^n0$ and $a^nx(xa^n - a^nx) = a^nx0$. Thus we have

$$(xa^n - a^nx)^2 = xa^n(xa^n - a^nx) - a^nx(xa^n - a^nx) = (xa^n - a^nx)0.$$

This equality implies that $xa^n - a^n x = 0$ by using Lemma 3, that is, $xa^n = a^n x$. Hence

$$a^n = xa^{2n} = (a^n x)a^n = (xa^n)a^n = xa^{2n}.$$

Consequently R is right π-regular. □ □

From Propositions 6 and 7, we obtain the following statement.

THEOREM 8. *The following statements are equivalent for any strongly reduced near-ring R:*

(1) R is a left π-regular near-ring.

(2) R is a strongly left π-regular near-ring.

(3) R is a strongly π-regular near-ring.

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