

HARMONIC TRANSFORMATIONS OF THE HYPERBOLIC PLANE

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ABSTRACT. Let (H, g) denote the upper half plane in R^2 with the Riemannian metric $g := ((dx)^2 + (dy)^2)/y^2$. First of all we get a necessary and sufficient condition for a diffeomorphism ϕ of (H, g) to be a harmonic map. And, we obtain the fact that if a diffeomorphism ϕ of (H, g) is a harmonic function, then the following facts are equivalent:

- (1) ϕ is a harmonic map;
- (2) ϕ is an affine transformation;
- (3) ϕ is an isometry (motion).

1. Introduction

Let H be the hyperbolic plane. Considering the points of H as all complex numbers with positive imaginary parts, we can view a diffeomorphism ϕ of H to be a complex-valued function defined on the upper half-plane H , and set $z' = \phi(z)$. If $\phi(z)$ is an analytic function of z , then ϕ is said to be a *holomorphic transformation* of H . Let $Hol(H)$ (resp. $Iso(H)$) be the group of all holomorphic transformations (resp. all motions (isometries)) of H . As is known in the theory of complex variables, the group $Hol(H)$ coincides with the group of all linear fractional transformations ([1, p. 182])

$$z(\in H) \longmapsto (az + b)/(cz + d) \in H, \text{ where } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(R).$$

On the other hand, $\theta : z \longmapsto -\bar{z}$ is a diffeomorphism of H such that $\theta \in Iso(H)$ and $\theta^2 = I$. Then, the following relation between $Iso(H)$ and $Hol(H)$ is well known.

Received September 04, 2009; Accepted November 19, 2009.

2000 Mathematics Subject Classification: Primary 53C07, 53C25.

Key words and phrases: harmonic map, Poincaré upper plane (hyperbolic plane).

THEOREM 1.1. *The group $Hol(H)$ is a normal subgroup of $Iso(H)$ of index 2, and $Iso(H) = Hol(H) + \theta Hol(H)$ holds.*

Let (M, g) be a Riemannian manifold, $Iso(M)$ (resp. $Aff(M)$) the group of all isometries (resp. affine transformations) of (M, g) . Then, $Iso(M) \subset Aff(M)$. Moreover, if $\phi : (M, g) \rightarrow (M, g)$ is an affine map, then ϕ is a harmonic map, but the converse does not hold, in general.

In this paper, we obtain a necessary and sufficient condition for a diffeomorphism ϕ of the Poincaré upper plane (hyperbolic plane) H to be a harmonic map (cf. Proposition 2.2). And then, we get the following main result (cf. Theorem 2.3).

THEOREM 1.2. *Let ϕ be a diffeomorphism of H . Assume ϕ is a harmonic function. Then, the following statements are equivalent*

- (i) ϕ is a harmonic map.
- (ii) ϕ is an affine transformation.
- (iii) ϕ is an isometry.

Here, we have the following question (cf. Remark 2.4):

Does there exist a harmonic transformation of (H, g) which is not isometric?

2. Harmonic transformations of the hyperbolic plane

Let $(M, g), (N, h)$ be two Riemannian manifolds. Let $\phi : M \rightarrow N$ be a smooth map. Let $E := \phi^{-1}TN$ be the induced bundle by ϕ over M of the tangent bundle TN of N . We denote by $\Gamma(E)$, the space of all sections V of E . We denote by $\nabla, {}^N\nabla$ the Levi-Civita connections of $(M, g), (N, h)$, respectively. Then we give the induced connection $\tilde{\nabla}$ on E by

$$(2.1) \quad (\tilde{\nabla}_X V)_x := \frac{d}{dt} {}^N P_{\phi(\gamma(t))}^{-1} V_{\gamma(t)}|_{t=0}, \quad X \in \Gamma(TM), \quad V \in \Gamma(E),$$

where $x \in M$, $\gamma(t)$ is a curve through x at $t = 0$ whose tangent vector at x is X_x , and ${}^N P_{\phi(\gamma(t))} : T_{\phi(x)}N \rightarrow T_{\phi(\gamma(t))}N$ is the parallel displacement along a curve $\phi(\gamma(s))$, $0 \leq s \leq t$, given by the Levi-Civita connection ${}^N\nabla$ of (N, h) .

If the tension field $\tau(\phi)$ of a smooth map ϕ of an m -dimensional Riemannian manifold (M, g) into another Riemannian manifold (N, h) vanishes on M , then ϕ is said to be a harmonic map (cf. [2, 9]) of (M, g) into (N, h) , where

$$(2.2) \quad \tau(\phi) := \sum_{i=1}^m (\tilde{\nabla}_{e_i} \phi_* e_i - \phi_* \nabla_{e_i} e_i),$$

for $\{e_i\}_{i=1}^m$ an (locally defined) orthonormal frame on (M, g) .

In general, the Riemannian connection ∇ for the Riemannian metric g on a Riemannian manifold (M, g) is given by

$$(2.3) \quad \begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) \\ &\quad - g(X, [Y, Z]) + g(Y, [Z, X]) + g(Z, [X, Y]) \end{aligned}$$

for $X, Y, Z \in \mathfrak{X}(M)$ (cf. [3]).

Let H denote the set of all points in the (x, y) -plane with positive y coordinates, and call it the upper half plane. Let $p = (x_o, y_o) \in H$, and let the components of $u, u' (\in T_p H)$ be $(\xi, \eta), (\xi', \eta')$, respectively. We define a Riemannian metric g on H by letting

$$(2.4) \quad g_p(u, u') = (\xi\xi' + \eta\eta')/y_o^2.$$

If we consider H to be the set of all complex numbers z with positive imaginary part $I(z)$, then we can write the Riemannian metric g on H as $dz d\bar{z}/I(z)^2$. The Riemannian manifold (H, g) is called the hyperbolic plane (Poincaré upper plane). Here

$$(2.5) \quad \left\{ y \frac{\partial}{\partial x} =: e_1, y \frac{\partial}{\partial y} =: e_2 \right\}$$

is an (globally defined) orthonormal frame on (H, g) . From (2,3) and (2,5), we get

$$(2.6) \quad \nabla_{e_1} e_1 = e_2, \quad \nabla_{e_1} e_2 = -e_1, \quad \nabla_{e_2} e_1 = \nabla_{e_2} e_2 = 0,$$

$$\nabla_{\frac{\partial}{\partial x}} \frac{\partial}{\partial x} = -\nabla_{\frac{\partial}{\partial y}} \frac{\partial}{\partial y} = (1/y) \frac{\partial}{\partial y}, \quad \nabla_{\frac{\partial}{\partial x}} \frac{\partial}{\partial y} = \nabla_{\frac{\partial}{\partial y}} \frac{\partial}{\partial x} = (-1/y) \frac{\partial}{\partial x},$$

where ∇ is the Levi-Civita connection for the metric g on (H, g) . Let ϕ be a diffeomorphism of (H, g) which is denoted by

$$(2.7) \quad \phi(x, y) = (u(x, y), v(x, y)) \in H, \quad ((x, y) \in H).$$

From (2.5) and (2.7), we obtain

$$(2.8) \quad \begin{cases} \phi_* e_1 = y \left(u_x \left(\frac{\partial}{\partial x} \right)_{(u,v)} + v_x \left(\frac{\partial}{\partial y} \right)_{(u,v)} \right), \\ \phi_* e_2 = y \left(u_y \left(\frac{\partial}{\partial x} \right)_{(u,v)} + v_y \left(\frac{\partial}{\partial y} \right)_{(u,v)} \right), \end{cases}$$

where ϕ_* is the differential map of ϕ , and $(u, v) := (u(x, y), v(x, y)) \in H$ $((x, y) \in H)$. And, from (2.6) and (2.8), we have

$$(2.9) \quad \begin{cases} \phi_*(\nabla_{e_2} e_1) = \phi_*(\nabla_{e_2} e_2) = 0, \\ \phi_*(\nabla_{e_1} e_1) = y \left(u_y \left(\frac{\partial}{\partial x} \right)_{(u,v)} + v_y \left(\frac{\partial}{\partial y} \right)_{(u,v)} \right), \\ \phi_*(\nabla_{e_1} e_2) = -y \left(u_x \left(\frac{\partial}{\partial x} \right)_{(u,v)} + v_x \left(\frac{\partial}{\partial y} \right)_{(u,v)} \right). \end{cases}$$

By virtue of (2.1), (2.5), (2.6) and (2.8), we get

$$(2.10) \quad \begin{aligned} \tilde{\nabla}_{e_1} \phi_* e_1 &= y^2 \left(u_{xx} - \frac{2u_x v_x}{v} \right) (\partial/\partial x)_{(u,v)} \\ &\quad + y^2 \left\{ v_{xx} + \frac{(u_x^2 - v_x^2)}{v} \right\} (\partial/\partial y)_{(u,v)}, \\ \tilde{\nabla}_{e_1} \phi_* e_2 &= y^2 \left\{ u_{xy} - \frac{(u_x v_y + u_y v_x)}{v} \right\} (\partial/\partial x)_{(u,v)} \\ &\quad + y^2 \left\{ v_{xy} + \frac{(u_x u_y - v_x v_y)}{v} \right\} (\partial/\partial y)_{(u,v)}, \\ \tilde{\nabla}_{e_2} \phi_* e_1 &= y \left\{ u_x + y u_{xy} - \frac{y(u_x v_y + u_y v_x)}{v} \right\} (\partial/\partial x)_{(u,v)} \\ &\quad + y \left\{ v_x + y v_{xy} + \frac{y(u_x u_y - v_x v_y)}{v} \right\} (\partial/\partial y)_{(u,v)}, \\ \tilde{\nabla}_{e_2} \phi_* e_2 &= y \left\{ u_y + y u_{yy} - \frac{2y u_y v_y}{v} \right\} (\partial/\partial x)_{(u,v)} \\ &\quad + y \left\{ v_y + y v_{yy} + \frac{y(u_y^2 - v_y^2)}{v} \right\} (\partial/\partial y)_{(u,v)}. \end{aligned}$$

Thus, from (2.9) and (2.10), we get

PROPOSITION 2.1. *Let ϕ be a diffeomorphism of (H, g) . Then, ϕ is an affine transformation if and only if*

$$(2.11) \quad \begin{aligned} u_{xx} - \frac{2u_x v_x}{v} &= - \left(u_{yy} - \frac{2u_y v_y}{v} \right) = \frac{u_y}{y}, \\ v_{xx} + \frac{(u_x^2 - v_x^2)}{v} &= - \left(v_{yy} + \frac{u_y^2 - v_y^2}{v} \right) = \frac{v_y}{y}, \\ u_{xy} - \frac{(u_x v_y + u_y v_x)}{v} &= \frac{-u_x}{y}, \\ v_{xy} + \frac{(u_x u_y - v_x v_y)}{v} &= \frac{-v_x}{y}. \end{aligned}$$

By the help of (2.2), (2.9) and (2.10), we get

PROPOSITION 2.2. *Let ϕ be a diffeomorphism of (H, g) . Then, a necessary and sufficient condition for the map ϕ to be a harmonic map is*

$$(2.12) \quad \begin{aligned} u_{xx} + u_{yy} - \frac{2(u_x v_x + u_y v_y)}{v} &= 0 \quad \text{and} \\ v_{xx} + v_{yy} + \frac{(u_x^2 + u_y^2 - v_x^2 - v_y^2)}{v} &= 0. \end{aligned}$$

Now, assume real functions u and v in a diffeomorphism ϕ of (H, g) such that $\phi(x, y) = (u(x, y), v(x, y)) \in H$ are harmonic functions, that is,

$$(2.13) \quad \Delta_g u = \Delta_g v = 0,$$

where $\Delta_g := \sum_{i=1}^2 (e_i^2 - \nabla_{e_i} e_i)$ is the Laplacian operator on (H, g) . A map ϕ satisfying (2.13) is said to be a *harmonic function*. Then, from (2.5), (2.6) and (2.13), we have

$$(2.14) \quad \Delta_g u = y^2(u_{xx} + u_{yy}) = 0, \quad \Delta_g v = y^2(v_{xx} + v_{yy}) = 0.$$

If $\phi : (H, g) \rightarrow (H, g)$ is a diffeomorphism, then we get at each point $p \in H$

$$(2.15) \quad u_x v_y - u_y v_x \neq 0.$$

From (2.15) and Proposition 2.2, we get the fact that a necessary and sufficient condition for a harmonic function ϕ , which is a diffeomorphism of (H, g) , to be a harmonic map is

$$(2.16) \quad u_x v_x + u_y v_y = 0 \quad \text{and} \quad u_x^2 + u_y^2 = v_x^2 + v_y^2.$$

For a diffeomorphism ϕ of (H, g) , (2.16) holds if and only if

$$(2.17) \quad \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}^{-1} = (u_x^2 + u_y^2)^{-1} \begin{pmatrix} u_x & v_x \\ u_y & v_y \end{pmatrix}$$

holds. By the help of (2.16) and (2.17), we obtain the fact that a necessary and sufficient condition for a harmonic function ϕ , being a diffeomorphism of (H, g) , to be a harmonic map is

$$(2.18) \quad u_x = \pm v_y \quad \text{and} \quad u_y = \mp v_x \quad (\text{double signs in the same order}).$$

By virtue of Theorem 1.1 and (2.18), we obtain the following

THEOREM 2.3. *Let ϕ be a diffeomorphism of (H, g) . Assume the map ϕ is a harmonic function. Then, the following statements are equivalent:*

- (1) *the map ϕ is a harmonic map.*
- (2) *ϕ is an affine transformation of (H, g) .*
- (3) *ϕ is an isometry (motion) of (H, g) .*

REMARK 2.4. We would like to know whether or not there exists a diffeomorphism ϕ of (H, G) which is not isometric, but satisfies the condition (2.12) appeared in the Proposition 2.2. In other words, the question is as follows:

Does there exist a harmonic transformation of (H, g) which is not isometric?

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