JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **22**, No. 4, December 2009

ON THE STABILITY OF AN AQCQ-FUNCTIONAL EQUATION

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Abstract. In this paper, we prove the generalized Hyers-Ulam stability of the following additive-quadratic-cubic-quartic functional equation

$$\begin{aligned}(0.1)\qquad f(x+2y)+f(x-2y) &= 4f(x+y)+4f(x-y)\\ &\quad -6f(x)+f(2y)+f(-2y)-4f(y)-4f(-y)\end{aligned}$$

in Banach spaces.

1. Introduction and preliminaries

The stability problem of functional equations originated from a question of Ulam [36] concerning the stability of group homomorphisms. Hyers [9] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [1] for additive mappings and by Th.M. Rassias [26] for linear mappings by considering an unbounded Cauchy difference. The paper of Th.M. Rassias [26] has provided a lot of influence in the development of what we call *generalized Hyers-Ulam stability* or as *Hyers-Ulam-Rassias stability* of functional equations. A generalization of the Th.M. Rassias theorem was obtained by Găvruta [8] by replacing the unbounded Cauchy difference by a general control function in the spirit of Th.M. Rassias' approach.

The functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y)$$

is called a *quadratic functional equation*. In particular, every solution of the quadratic functional equation is said to be a *quadratic mapping*. A

Received September 02, 2009; Accepted November 06, 2009.
2000 Mathematics Subject Classification: Primary 39B72, 54E40.
Key words and phrases: Banach space, generalized Hyers-Ulam stability, additive-quadratic-cubic-quartic functional equation.
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generalized Hyers-Ulam stability problem for the quadratic functional equation was proved by Skof [35] for mappings $f : X \to Y$, where X is a normed space and Y is a Banach space. Cholewa [2] noticed that the theorem of Skof is still true if the relevant domain X is replaced by an Abelian group. Czerwik [3] proved the generalized Hyers-Ulam stability of the quadratic functional equation. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [10], [13], [19]–[22], [23]–[34]).

In [12], Jun and Kim considered the following cubic functional equation

$$f(2x+y)+f(2x-y)=2f(x+y)+2f(x-y)+12f(x). \tag{1.1}$$

It is easy to show that the function $f(x) = x^3$ satisfies the functional equation (1.1), which is called a *cubic functional equation* and every solution of the cubic functional equation is said to be a *cubic mapping.*

In [14], Lee et al. considered the following quartic functional equation

$$\begin{aligned} &f(2x+y)+f(2x-y) \\ &\qquad = 4f(x+y)+4f(x-y)+24f(x)-6f(y). \end{aligned} \tag{1.2}$$

It is easy to show that the function $f(x) = x^4$ satisfies the functional equation (1.2), which is called a *quartic functional equation* and every solution of the quartic functional equation is said to be a *quartic mapping.*

This paper is organized as follows: In Section 2, we prove the generalized Hyers-Ulam stability of the additive-quadratic-cubic-quartic functional equation (0.1) in Banach spaces for an odd case. In Section 3, we prove the generalized Hyers-Ulam stability of the additive-quadratic-cubic-quartic functional equation (0.1) in Banach spaces for an even case.

Throughout this paper, assume that X is a normed vector space and that Y is a Banach space.

2. Generalized Hyers-Ulam stability of the functional equation (0.1): an odd case

One can easily show that an odd mapping $f : X \to Y$ satisfies (0.1) if and only if the odd mapping mapping $f : X \to Y$ is an additive-cubic mapping, i.e.,

$$f(x+2y)+f(x-2y)=4f(x+y)+4f(x-y)-6f(x).$$

It was shown in Lemma 2.2 of [6] that $g(x) := f(2x)-2f(x)$ and $h(x) := f(2x) - 8f(x)$ are cubic and additive, respectively, and that $f(x) = \frac{1}{6}g(x) - \frac{1}{6}h(x)$.

One can easily show that an even mapping $f : X \to Y$ satisfies (0.1) if and only if the even mapping $f : X \to Y$ is a quadratic-quartic mapping, i.e.,

$$f(x+2y)+f(x-2y) = 4f(x+y)+4f(x-y)-6f(x)+2f(2y)-8f(y).$$

It was shown in Lemma 2.1 of [5] that $g(x) := f(2x)-4f(x)$ and $h(x) := f(2x)-16f(x)$ are quartic and quadratic, respectively, and that $f(x) = \frac{1}{12}g(x) - \frac{1}{12}h(x)$.

For a given mapping $f : X \to Y$, we define

$$\begin{aligned} Df(x,y) :&= f(x+2y)+f(x-2y)-4f(x+y)-4f(x-y)+6f(x) \\ &- f(2y)-f(-2y)+4f(y)+4f(-y) \end{aligned}$$

for all $x, y \in X$.

We prove the generalized Hyers-Ulam stability of the functional equation $Df(x,y) = 0$ in Banach spaces: an odd case.

Theorem 2.1. *Let $\varphi : X^2 \to [0,\infty)$ be a function such that*

$$\Phi(x,y) := \sum_{n=0}^{\infty} 8^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) < \infty \tag{2.1}$$

for all $x, y \in X$. Let $f : X \to Y$ be an odd mapping satisfying

$$\|Df(x,y)\| \le \varphi(x,y) \tag{2.2}$$

for all $x, y \in X$. Then there exists a unique cubic mapping $C : X \to Y$ such that

$$\|f(2x)-2f(x)-C(x)\| \le 4\Phi\left(\frac{x}{2},\frac{x}{2}\right) + \Phi\left(x,\frac{x}{2}\right) \tag{2.3}$$

for all $x \in X$.

Proof. Letting $x = y$ in (2.2), we get

$$\|f(3y)-4f(2y)+5f(y)\| \le \varphi(y,y) \tag{2.4}$$

for all $y \in X$.

Replacing x by $2y$ in (2.2), we get

$$\|f(4y)-4f(3y)+6f(2y)-4f(y)\| \le \varphi(2y,y) \tag{2.5}$$

for all $y \in X$.

By (2.4) and (2.5),

$$
\begin{aligned}
\|f(4y) - 10f(2y) &+ 16f(y)\| \leq \|4(f(3y) - 4f(2y) + 5f(y))\| \\
&+ \|f(4y) - 4f(3y) + 6f(2y) - 4f(y)\| \\
&\leq 4\varphi(y, y) + \varphi(2y, y)
\end{aligned}
\tag{2.6}
$$

for all $y \in X$. Letting $y := \frac{x}{2}$ and $g(x) := f(2x) - 2f(x)$ for all $x \in X$, we get

$$
\left\| g(x) - 8g\left(\frac{x}{2}\right) \right\| \leq 4\varphi\left(\frac{x}{2}, \frac{x}{2}\right) + \varphi\left(x, \frac{x}{2}\right) \tag{2.7}
$$

for all $x \in X$. Hence

$$
\begin{aligned}
&\|8^l g(\frac{x}{2^l}) - 8^m g(\frac{x}{2^m})\| \\
&\leq \sum_{j=l}^{m-1} 4 \cdot 8^j \varphi\left(\frac{x}{2^{j+1}}, \frac{x}{2^{j+1}}\right) + \sum_{j=l}^{m-1} 8^j \varphi\left(\frac{x}{2^j}, \frac{x}{2^{j+1}}\right)
\end{aligned}
\tag{2.8}
$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (2.1) and (2.8) that the sequence $\{8^k g(\frac{x}{2^k})\}$ is Cauchy for all $x \in X$. Since Y is complete, the sequence $\{8^k g(\frac{x}{2^k})\}$ converges. So one can define the mapping $C : X \to Y$ by

$$
C(x) := \lim_{k\to\infty} 8^k g\left(\frac{x}{2^k}\right)
$$

for all $x \in X$.

By (2.1) and (2.2),

$$
\begin{aligned}
\|DC(x, y)\| &= \lim_{k\to\infty} 8^k \left\| Dg\left(\frac{x}{2^k}, \frac{y}{2^k}\right) \right\| \\
&\leq \lim_{k\to\infty} 8^k \left(\varphi\left(\frac{2x}{2^k}, \frac{2y}{2^k}\right) + 2\varphi\left(\frac{x}{2^k}, \frac{y}{2^k}\right) \right) = 0
\end{aligned}
$$

for all $x, y \in X$. So $DC(x, y) = 0$. Since $g : X \to Y$ is odd, $C : X \to Y$ is odd. So the mapping $C : X \to Y$ is cubic. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.8), we get (2.3). So there exists a cubic mapping $C : X \to Y$ satisfying (2.3).

Now, let $C' : X \to Y$ be another cubic mapping satisfying (2.3). Then we have

$$
\begin{aligned}
\|C(x) - C'(x)\| &= 8^q \left\| C\left(\frac{x}{2^q}\right) - C'\left(\frac{x}{2^q}\right) \right\| \\
&\leq 8^q \left\| C\left(\frac{x}{2^q}\right) - g\left(\frac{x}{2^q}\right) \right\| + 8^q \left\| C'\left(\frac{x}{2^q}\right) - g\left(\frac{x}{2^q}\right) \right\| \\
&\leq 2 \cdot 4 \cdot 8^q \Phi\left(\frac{x}{2^{q+1}}, \frac{x}{2^{q+1}}\right) + 2 \cdot 8^q \Phi\left(\frac{x}{2^q}, \frac{x}{2^{q+1}}\right),
\end{aligned}
$$

which tends to zero as $q \to \infty$ for all $x \in X$. So we can conclude that $C(x) = C'(x)$ for all $x \in X$. This proves the uniqueness of C. □

Corollary 2.2. *Let $\theta \geq 0$ and let p be a real number with $p > 3$. Let $f : X \to Y$ be an odd mapping satisfying*

$$\|Df(x,y)\| \leq \theta(\|x\|^p + \|y\|^p) \tag{2.9}$$

for all $x, y \in X$. Then there exists a unique cubic mapping $C : X \to Y$ such that

$$\|f(2x) - 2f(x) - C(x)\| \leq \frac{2^p + 9}{2^p - 8}\theta\|x\|^p$$

for all $x \in X$.

Proof. The proof follows from Theorem 2.1 by taking

$$\varphi(x,y) := \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in X$. □

Theorem 2.3. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that*

$$\Phi(x,y) := \sum_{n=0}^{\infty} \frac{1}{8^n}\varphi\left(2^n x, 2^n y\right) < \infty$$

for all $x, y \in X$. Let $f : X \to Y$ be an odd mapping satisfying (2.2). Then there exists a unique cubic mapping $C : X \to Y$ such that

$$\|f(2x) - 2f(x) - C(x)\| \leq \frac{1}{2}\Phi\left(x, x\right) + \frac{1}{8}\Phi\left(2x, x\right)$$

for all $x \in X$.

Proof. It follows from (2.7) that

$$\left\|g(x) - \frac{1}{8}g\left(2x\right)\right\| \leq \frac{1}{2}\varphi\left(x, x\right) + \frac{1}{8}\varphi\left(2x, x\right)$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 2.1. □

Corollary 2.4. *Let $\theta \geq 0$ and let p be a real number with $0 < p < 3$. Let $f : X \to Y$ be an odd mapping satisfying (2.9). Then there exists a unique cubic mapping $C : X \to Y$ such that*

$$\|f(2x) - 2f(x) - C(x)\| \leq \frac{9 + 2^p}{8 - 2^p}\theta\|x\|^p$$

for all $x \in X$.

Proof. The proof follows from Theorem 2.3 by taking

$$\varphi(x,y) := \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in X$. □

THEOREM 2.5. *Let $\varphi : X^2 \to [0,\infty)$ be a function such that*

$$\Phi(x,y) := \sum_{n=0}^{\infty} 2^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) < \infty$$

for all $x, y \in X$. Let $f : X \to Y$ be an odd mapping satisfying (2.2). *Then there exists a unique additive mapping $A : X \to Y$ such that*

$$\|f(2x) - 8f(x) - A(x)\| \le 4\Phi\left(\frac{x}{2}, \frac{x}{2}\right) + \Phi\left(x, \frac{x}{2}\right)$$

for all $x \in X$.

Proof. Letting $y := \frac{x}{2}$ and $g(x) := f(2x) - 8f(x)$ in (2.6), we get

$$\left\|g(x) - 2g\left(\frac{x}{2}\right)\right\| \le 4\varphi\left(\frac{x}{2}, \frac{x}{2}\right) + \varphi\left(x, \frac{x}{2}\right) \tag{2.10}$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 2.1. □

COROLLARY 2.6. *Let $\theta \ge 0$ and let p be a real number with $p > 1$. Let $f : X \to Y$ be an odd mapping satisfying* (2.9). *Then there exists a unique additive mapping $A : X \to Y$ such that*

$$\|f(2x) - 8f(x) - A(x)\| \le \frac{2^p + 9}{2^p - 2}\theta\|x\|^p$$

for all $x \in X$.

Proof. The proof follows from Theorem 2.5 by taking

$$\varphi(x,y) := \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in X$. □

THEOREM 2.7. *Let $\varphi : X^2 \to [0,\infty)$ be a function such that*

$$\Phi(x,y) := \sum_{n=0}^{\infty} \frac{1}{2^n} \varphi\left(2^n x, 2^n y\right) < \infty \tag{2.11}$$

for all $x, y \in X$. Let $f : X \to Y$ be an odd mapping satisfying (2.2). *Then there exists a unique additive mapping $A : X \to Y$ such that*

$$\|f(2x) - 8f(x) - A(x)\| \le 2\Phi\left(x, x\right) + \frac{1}{2}\Phi\left(2x, x\right)$$

for all $x \in X$.

Proof. It follows from (2.10) that

$$\left\| g(x) - \frac{1}{2} g\left(2x\right) \right\| \le 2\varphi\left(x, x\right) + \frac{1}{2}\varphi\left(2x, x\right)$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 2.1. □

Corollary 2.8. *Let $\theta \ge 0$ and let p be a real number with $0 < p < 1$. Let $f : X \to Y$ be an odd mapping satisfying (2.9). Then there exists a unique additive mapping $A : X \to Y$ such that*

$$\|f(2x) - 8f(x) - A(x)\| \le \frac{9 + 2^p}{2 - 2^p} \theta \|x\|^p$$

for all $x \in X$.

Proof. The proof follows from Theorem 2.7 by taking

$$\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in X$. □

3. Generalized Hyers-Ulam stability of the functional equation (0.1): an even case

We prove the generalized Hyers-Ulam stability of the functional equation $Df(x, y) = 0$ in Banach spaces: an even case.

Theorem 3.1. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that*

$$\Psi(x, y) := \sum_{n=0}^{\infty} 16^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) < \infty \tag{3.1}$$

for all $x, y \in X$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (2.2). Then there exists a unique quartic mapping $Q : X \to Y$ such that

$$\|f(2x) - 4f(x) - Q(x)\| \le 4\Psi\left(\frac{x}{2}, \frac{x}{2}\right) + \Psi\left(x, \frac{x}{2}\right)$$

for all $x \in X$.

Proof. Letting $x = y$ in (2.2), we get

$$\|f(3y) - 6f(2y) + 15f(y)\| \le \varphi(y, y) \tag{3.2}$$

for all $y \in X$.

Replacing x by $2y$ in (2.2), we get

$$\|f(4y) - 4f(3y) + 4f(2y) + 4f(y)\| \le \varphi(2y, y) \tag{3.3}$$

for all $y \in X$.

By (3.2) and (3.3),

$$\begin{aligned} &\|f(4x) - 20f(2x) + 64f(x)\| \\ &\quad \leq \|4(f(3x) - 6f(2x) + 15f(x))\| \\ &\quad +\|f(4x) - 4f(3x) + 4f(2x) + 4f(x)\| \\ &\quad \leq 4\varphi(x,x) + \varphi(2x,x) \end{aligned} \tag{3.4}$$

for all $x \in X$. Letting $g(x) := f(2x) - 4f(x)$ for all $x \in X$, we get

$$\left\|g(x) - 16g\left(\frac{x}{2}\right)\right\| \leq 4\varphi\left(\frac{x}{2}, \frac{x}{2}\right) + \varphi\left(x, \frac{x}{2}\right) \tag{3.5}$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 2.1. □

Corollary 3.2. *Let $\theta \geq 0$ and let p be a real number with $p > 4$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (2.9). Then there exists a unique quartic mapping $Q : X \to Y$ such that*

$$\|f(2x) - 4f(x) - Q(x)\| \leq \frac{2^p + 9}{2^p - 16}\theta\|x\|^p$$

for all $x \in X$.

Proof. The proof follows from Theorem 3.1 by taking

$$\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in X$. □

Theorem 3.3. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that*

$$\Psi(x, y) := \sum_{n=0}^{\infty} \frac{1}{16^n}\varphi\left(2^n x, 2^n y\right) < \infty$$

for all $x, y \in X$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (2.2). Then there exists a unique quartic mapping $Q : X \to Y$ such that

$$\|f(2x) - 4f(x) - Q(x)\| \leq \frac{1}{4}\Psi\left(x, x\right) + \frac{1}{16}\Psi\left(2x, x\right)$$

for all $x \in X$.

Proof. It follows from (3.5) that

$$\left\|g(x) - \frac{1}{16}g\left(2x\right)\right\| \leq \frac{1}{4}\varphi\left(x, x\right) + \frac{1}{16}\varphi\left(2x, x\right)$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 2.1. □

Corollary 3.4. *Let $\theta \geq 0$ and let p be a real number with $0 < p < 4$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (2.9). Then there exists a unique quartic mapping $Q : X \to Y$ such that*

$$\|f(2x) - 4f(x) - Q(x)\| \leq \frac{9 + 2^p}{16 - 2^p}\theta\|x\|^p$$

for all $x \in X$.

Proof. The proof follows from Theorem 3.3 by taking

$$\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in X$. □

Theorem 3.5. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that*

$$\Psi(x, y) := \sum_{n=0}^{\infty} 4^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) < \infty$$

for all $x, y \in X$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (2.2). Then there exists a unique quadratic mapping $T : X \to Y$ such that

$$\|f(2x) - 16f(x) - T(x)\| \leq 4\Psi\left(\frac{x}{2}, \frac{x}{2}\right) + \Psi\left(x, \frac{x}{2}\right)$$

for all $x \in X$.

Proof. Letting $g(x) := f(2x) - 16f(x)$ in (3.4), we get

$$\left\|g(x) - 4g\left(\frac{x}{2}\right)\right\| \leq 4\varphi\left(\frac{x}{2}, \frac{x}{2}\right) + \varphi\left(x, \frac{x}{2}\right) \tag{3.6}$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 2.1. □

Corollary 3.6. *Let $\theta \geq 0$ and let p be a real number with $p > 2$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (2.9). Then there exists a unique quadratic mapping $T : X \to Y$ such that*

$$\|f(2x) - 16f(x) - T(x)\| \leq \frac{2^p + 9}{2^p - 4}\theta\|x\|^p$$

for all $x \in X$.

Proof. The proof follows from Theorem 3.5 by taking

$$\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in X$. □

THEOREM 3.7. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that*

$$\Psi(x, y) := \sum_{n=0}^{\infty} \frac{1}{4^n} \varphi\left(2^n x, 2^n y\right) < \infty$$

for all $x, y \in X$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (2.2). Then there exists a unique quadratic mapping $T : X \to Y$ such that

$$\|f(2x) - 16f(x) - T(x)\| \le \Psi\left(x, x\right) + \frac{1}{4}\Psi\left(2x, x\right)$$

for all $x \in X$.

Proof. It follows from (3.6) that

$$\left\|g(x) - \frac{1}{4}g\left(2x\right)\right\| \le \varphi\left(x, x\right) + \frac{1}{4}\varphi\left(2x, x\right)$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 2.1. □

COROLLARY 3.8. *Let $\theta \ge 0$ and let p be a real number with $0 < p < 2$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (2.9). Then there exists a unique quadratic mapping $T : X \to Y$ such that*

$$\|f(2x) - 16f(x) - T(x)\| \le \frac{9 + 2^p}{4 - 2^p}\theta\|x\|^p$$

for all $x \in X$.

Proof. The proof follows from Theorem 3.7 by taking

$$\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in X$. □

Let $f_o(x) := \frac{f(x)-f(-x)}{2}$ and $f_e(x) := \frac{f(x)+f(-x)}{2}$. Then f_o is odd and f_e is even. f_o and f_e satisfy the functional equation (0.1). Let $g_o(x) := f_o(2x) - 2f_o(x)$ and $h_o(x) := f_o(2x) - 8f_o(x)$. Then $f_o(x) = \frac{1}{6}g_o(x) - \frac{1}{6}h_o(x)$. Let $g_e(x) := f_e(2x) - 4f_e(x)$ and $h_e(x) := f_e(2x) - 16f_e(x)$. Then $f_e(x) = \frac{1}{12}g_e(x) - \frac{1}{12}h_e(x)$. Thus

$$f(x) = \frac{1}{6}g_o(x) - \frac{1}{6}h_o(x) + \frac{1}{12}g_e(x) - \frac{1}{12}h_e(x).$$

So we obtain the following results.

Theorem 3.9. *Let $\varphi : X^2 \to [0,\infty)$ be a function satisfying* (3.1)*. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and* (2.2)*. Then there exist an additive mapping $A : X \to Y$, a quadratic mapping $T : X \to Y$, a cubic mapping $C : X \to Y$ and a quartic mapping $Q : X \to Y$ such that*

$$\begin{aligned}&\left\|f(x) - \frac{1}{6}A(x) - \frac{1}{12}T(x) - \frac{1}{6}C(x) - \frac{1}{12}Q(x)\right\|\\ &\le \frac{2}{3}\Phi_1\left(\frac{x}{2},\frac{x}{2}\right) + \frac{1}{6}\Phi_1\left(x,\frac{x}{2}\right) + \frac{1}{3}\Psi_2\left(\frac{x}{2},\frac{x}{2}\right) + \frac{1}{12}\Psi_2\left(x,\frac{x}{2}\right)\\ &+\frac{2}{3}\Phi_3\left(\frac{x}{2},\frac{x}{2}\right) + \frac{1}{6}\Phi_3\left(x,\frac{x}{2}\right) + \frac{1}{3}\Psi_4\left(\frac{x}{2},\frac{x}{2}\right) + \frac{1}{12}\Psi_4\left(x,\frac{x}{2}\right)\end{aligned}$$

for all $x \in X$. Here $\Phi_1 := \Phi, \Psi_2 := \Psi, \Phi_3 := \Phi$ and $\Psi_4 := \Psi$ are given in the statements of Theorems 2.5, 3.5, 2.1 and 3.1, respectively.

Corollary 3.10. *Let $\theta \ge 0$ and let p be a real number with $p > 4$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and* (2.9)*. Then there exist an additive mapping $A : X \to Y$, a quadratic mapping $T : X \to Y$, a cubic mapping $C : X \to Y$ and a quartic mapping $Q : X \to Y$ such that*

$$\begin{aligned}&\left\|f(x) - \frac{1}{6}A(x) - \frac{1}{12}T(x) - \frac{1}{6}C(x) - \frac{1}{12}Q(x)\right\|\\ &\le \left(\frac{2^p+9}{6(2^p-2)} + \frac{2^p+9}{12(2^p-4)} + \frac{2^p+9}{6(2^p-8)} + \frac{2^p+9}{12(2^p-16)}\right)\theta\|x\|^p\end{aligned}$$

for all $x \in X$.

Proof. The proof follows from Theorem 3.9 by taking

$$\varphi(x,y) := \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in X$. □

Theorem 3.11. *Let $\varphi : X^2 \to [0,\infty)$ be a function satisfying* (2.11)*. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and* (2.2)*. Then there exist an additive mapping $A : X \to Y$, a quadratic mapping $T : X \to Y$, a cubic mapping $C : X \to Y$ and a quartic mapping $Q : X \to Y$ such that*

$$\begin{aligned}&\left\|f(x) - \frac{1}{6}A(x) - \frac{1}{12}T(x) - \frac{1}{6}C(x) - \frac{1}{12}Q(x)\right\|\\ &\le \frac{1}{3}\Phi_1(x,x) + \frac{1}{12}\Phi_1(2x,x) + \frac{1}{12}\Psi_2(x,x) + \frac{1}{48}\Psi_2(2x,x)\\ &+\frac{1}{12}\Phi_3(x,x) + \frac{1}{48}\Phi_3(2x,x) + \frac{1}{48}\Psi_4(x,x) + \frac{1}{192}\Psi_4(2x,x)\end{aligned}$$

for all $x \in X$. *Here* $\Phi_1 := \Phi, \Psi_2 := \Psi, \Phi_3 := \Phi$ *and* $\Psi_4 := \Psi$ *are given in the statements of Theorems 2.7, 3.7, 2.3 and 3.3, respectively.*

Corollary 3.12. *Let* $\theta \geq 0$ *and let* p *be a real number with* $0 < p < 1$. *Let* $f : X \to Y$ *be a mapping satisfying* $f(0) = 0$ *and* (2.9). *Then there exist an additive mapping* $A : X \to Y$, *a quadratic mapping* $T : X \to Y$, *a cubic mapping* $C : X \to Y$ *and a quartic mapping* $Q : X \to Y$ *such that*

$$\begin{aligned}&\left\| f(x) - \frac{1}{6}A(x) - \frac{1}{12}T(x) - \frac{1}{6}C(x) - \frac{1}{12}Q(x) \right\| \\ &\quad \leq \left(\frac{2^p+9}{6(2-2^p)} + \frac{2^p+9}{12(4-2^p)} + \frac{2^p+9}{6(8-2^p)} + \frac{2^p+9}{12(16-2^p)} \right) \theta \|x\|^p\end{aligned}$$

for all $x \in X$.

Proof. The proof follows from Theorem 3.11 by taking

$$\varphi(x,y) := \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in X$. □

Acknowledgments. This work was supported by R&E Program in 2009.

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