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ON CERTAIN POLYNOMIALS HAVING ALL THEIR ZEROS EXCEPT FOR 1 ON A CIRCLE OF RADIUS < 1

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Abstract. Given $\alpha > 1$, there exist $C(1/\alpha)-$polynomials of the form $z^n - \sum_{k=0}^{n-1} a_k z^k$, where $\sum_{k=0}^{n-1} a_k = 1$, $a_{n-1} > 0$ and $a_k \geq 0$ for each k. In this paper, we obtain lower bounds for a_{n-1}.

1. Introduction

Throughout this note, n is an integer ≥ 3, $\alpha > 1$, and we denote $C(r)$ by the circle of radius r with center the origin.

If z is a complex number inside $C(1)$ which is not a positive real number, then there is an integer n such that z^n is a convex combination of lower integral powers $\{z^k : 0 \leq k < n\}$. Moreover the convex hull of the sequence $1, z, z^2, z^3, \cdots$ is a polygon; if n is the number of vertices of this polygon, then these vertices are precisely the first n powers of z. For the proofs of the above, see Lemma 2.1 and Theorem 2.2 of [1]. Conversely, if

$$z^n = \sum_{k=0}^{n-1} a_k z^k, \tag{1}$$

where $\sum_{k=0}^{n-1} a_k = 1$, $a_k \geq 0$ for each k, then it follows from Eneström-Kakeya theorem (see p. 136 of [2] for the statement and its proof) to

$$\frac{z^n - \displaystyle\sum_{k=0}^{n-1} a_k z^k}{z - 1}$$

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that all zeros of (1) do not lie outside $C(1)$. More precisely, the zeros of (1) are strictly inside $C(1)$ except for $z = 1$ since the average of points on $C(1)$ is strictly inside $C(1)$ unless all of the points are equal.

Whether or not certain polynomials have all their zeros on a circle is one of the most fundamental questions in the theory of distribution of polynomial zeros. Hence, in this paper, we study polynomials of type (1), $f(z) = z^n - \sum_{k=0}^{n-1} a_k z^k$, whose all zeros except for $z = 1$ lie on $C(1/\alpha)$. For convenience, we call these polynomials $C(1/\alpha)-$polynomials, and $\sum_{k=0}^{n-1} a_k z^k$ in $C(1/\alpha)-$polynomials their weighted sums, respectively.

By estimating some coefficients of lacunary polynomials, Kim [3] obtained sufficient conditions for nonexistence of lacunary $C(1/\alpha)-$polynomials whose the degree of weighted sums is $n-2$ and $n-3$, respectively. In particular, Kim's sufficient condition for the degree of weighted sum $n-2$ was best possible in certain senses. However, in the study of the case with the degree of weighted sum $n-1$, Kim [3] showed that, given $\alpha > 1$, there always exist $C(1/\alpha)-$polynomials whose degree of weighted sum is $n-1$. The purpose of this note is to obtain following lower bounds for a_{n-1} for existence of such $C(1/\alpha)-$polynomials.

THEOREM 1.1. *If $f(z) = z^n - \sum_{k=0}^{n-1} a_k z^k$ is $C(1/\alpha)-$polynomial where $\sum_{k=0}^{n-1} a_k = 1$, $a_{n-1} > 0$ and $a_k \geq 0$ for each k, then, for $p, q > 0$, we have*

$$a_{n-1} \geq 1 - \frac{n-1}{(1-1/\alpha^p)^{1/q}} \frac{p^{1/q} q^{1/p}}{(p+q)^{1/p+1/q}}.$$

2. Proof and example

Proof of Theorem 1.1. Let

$$f(z) = z^n - \sum_{k=0}^{n-1} a_k z^k = (z - z_1) \prod_{k=2}^{n} (z - z_k)$$

is $C(1/\alpha)-$polynomial where $z_1 = 1$, $\sum_{k=0}^{n-1} a_k = 1$, $a_{n-1} > 0$ and $a_k \geq 0$ for each k. For p, $q > 0$, it follows from the identity

$$\sup_{0 \leq a \leq 1} a^q (1 - a^p) = \frac{p}{q} \left(\frac{q}{p+q} \right)^{1+q/p}$$

and $\alpha > 1$ that

$$\frac{1}{\alpha^q}\left(1-\frac{1}{\alpha^p}\right) \le \frac{p}{q}\left(\frac{q}{p+q}\right)^{1+q/p}$$

or

$$\frac{1}{1-1/\alpha^p} \ge \frac{1}{\alpha^q}\frac{q}{p}\left(\frac{p+q}{q}\right)^{1+q/p} = \frac{(p+q)^{1+q/p}}{pq^{q/p}}\frac{1}{\alpha^q}$$

or

$$\frac{1}{(1-1/\alpha^p)^{1/q}} \ge \frac{(p+q)^{1/p+1/q}}{p^{1/q}q^{1/p}}\frac{1}{\alpha}.$$

Since $z_1 = 1$ and $|z_k| = 1/\alpha$ for $2 \le k \le n$, we have

$$\begin{aligned}\frac{n-1}{(1-1/\alpha^p)^{1/q}} &\ge \frac{(p+q)^{1/p+1/q}}{p^{1/q}q^{1/p}}\sum_{k=2}^{n}|z_k| \\ &\ge \frac{(p+q)^{1/p+1/q}}{p^{1/q}q^{1/p}}\left|\sum_{k=2}^{n} z_k\right| \\ &= \frac{(p+q)^{1/p+1/q}}{p^{1/q}q^{1/p}}(1-a_{n-1}),\end{aligned}$$

which proves the theorem. □

Example 2.1. Suppose that $f(z) = z^7 - \sum_{k=0}^{6} a_k z^k$ is $C(1/20)-$polynomial where $\sum_{k=0}^{6} a_k = 1$, $a_6 > 0$ and $a_k \ge 0$ for each k. Then, by choosing $p = q = 1/2$, computer algebra suggests $a_6 \ge 0.377889\cdots$.

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