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MORPHISMS BETWEEN FANO MANIFOLDS GIVEN BY COMPLETE INTERSECTIONS

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Abstract. We study the existence of surjective morphisms between Fano manifolds of Picard number 1, when the source is given by the intersection of a cubic hypersurface and either a quadric or another cubic hypersurface in a projective space.

1. Introduction

For two Fano manifolds X and Y of Picard number 1, one may ask if there is a surjective morphism

$$f : X \to Y.$$

In particular, one may ask if a Fano manifold X of Picard number 1 admits a surjective endomorphism of degree bigger than 1. On these questions, there have been several attempts to confirm the following conjectures. Recall that the *index* $i(X)$ of X is defined by the number i such that

$$-K_X \cong \mathcal{O}_X(i),$$

where $\mathcal{O}_X(1)$ is the ample generator of $\mathrm{Pic}(X)$.

Conjecture 1.1. (Peternell, [6]) *If there is a surjective morphism*

$$f : X \to Y$$

between Fano manifolds X and Y of Picard number 1, then

$$i(X) \leq i(Y).$$

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Conjecture 1.2. (See [1], Conjecture 1.1) *If a Fano manifold X of Picard number 1 admits a surjective endomorphism of degree > 1, then $X \cong \mathbb{P}^n$.* □

Beauville observed that the Chern number inequality devised by Amerik, Rovinsky, and Van de Ven [2] can be used to prove the following.

Proposition 1.3. ([3]) *A smooth hypersurface in $\mathbb{P}^{n+1}$ of degree d admits no endomorphisms of degree bigger than 1 if $n \geq 2$ and $d \geq 3$.* □

It has been observed that the same Chern number inequality can actually be applied to more general situations.

Proposition 1.4. ([4]) *Let X and Y be smooth Fano hypersurfaces in $\mathbb{P}^{n+1}$, $n \geq 3$, of degree d_X and d_Y respectively, $d_X, d_Y \geq 3$. If there is a surjective morphism*

$$f : X \to Y,$$

then either $i_X < i_Y$ or f is an isomorphism. □

Proposition 1.5. ([5], Theorem 1.5) *Let V be a Fano manifold of Picard number 1. Assume that $\dim V \geq 4$ and the ample generator $\mathcal{O}_V(1)$ of $\mathrm{Pic}(V)$ is very ample. If X is a smooth hypersurface of V cut out by a member of $|\mathcal{O}_V(d)|$, $d \geq 4$, then X admits no endomorphism of degree bigger than 1.* □

But note here that we need the conditions "$d \geq 3$" in Theorem 1.3, "$d_X, d_Y \geq 3$" in Theorem 1.4, and "$d \geq 4$" in Theorem 1.5. This kind of degree condition is required simply because the Chern number inequality, which is the main tool of proof, does not work effectively when $d \leq 2$.

A natural question in this direction are the following:

- Can Propositions 1.4 and 1.5 be generalized to the case of complete intersections?
- Is the statement of Proposition 1.5 still true for the case when $d = 3$?

The first question for Proposition 1.5 can be answered affirmatively if it is a complete intersection of k hypersurfaces of degree $d_1, d_2, \cdots, d_k$ where $\max\{d_1, \cdots, d_k\} \geq 4$: see Theorem 3.1 of [5].

In this paper, we consider the cases of (1) the intersection of a quadric and a cubic hypersurface and (2) the intersection of two cubic hypersurfaces in a projective space. For $n \geq 3$ (resp. $n \geq 4$), projective manifolds given by the intersection of a smooth quadric and a smooth cubic hypersurface (resp. of two smooth cubic hypersurfaces) in $\mathbb{P}^{n+2}$ are Fano manifolds of Picard number 1. We prove the following.

Theorem 1.6. *Let $\tilde{X}$ be either a smooth quadric or a smooth cubic hypersurface in $\mathbb{P}^{n+2}$, $n \geq 3$. Let X be a submanifold of dimension n given by the intersection of $\tilde{X}$ and a cubic hypersurface in $\mathbb{P}^{n+2}$. Let Y be a smooth subvariety of $\mathbb{P}^{n+2}$ cut out by two smooth hypersurfaces of degree d_1 and d_2 respectively. If there is a surjective morphism*

$$f : X \to Y$$

of degree bigger than 1, then either $d_1 = d_2 = 2$ or $d_1 d_2 \leq 3$. □

Note that $i(Y) = n + 3 - (d_1 + d_2)$. Hence the above result implies:
(i) X admits no endomorphisms of degree bigger than 1, and
(ii) if there is a surjective morphism $f : X \to Y$ where Y is a smooth subvariety of $\mathbb{P}^{n+2}$ cut out by two smooth hypersurfaces, then either $i(X) < i(Y)$ or f is an isomorphism.

In particular, we get the following.

Corollary 1.7. *Let X be a smooth subvariety of $\mathbb{P}^{n+2}$ cut out by two smooth hypersurfaces of degree d_1 and d_2 respectively, where $d_1 \geq d_2$. If $n \geq 3$ and $d_1 \geq 3$, then X does not admit an endomorphism of degree bigger than 1.*

Proof. When $d_1 \geq 4$, this is a special case of Theorem 3.1 of [5]. When $d_2 = 1$, X is a hypersurface of $\mathbb{P}^{n+1}$ of degree $d_1 \geq 3$ and the wanted result was shown in [3]. For the remaining cases where $d_1 = 3$ and $d_2 \geq 2$, the wanted result follows from Theorem 1.6. □

2. Chern number inequalities

First let us recall the inequality proven by Amerik, Rovinsky and Van de Ven.

Lemma 2.1. ([2], Corollary 1.2) *Let $f : X \to Y$ be a finite morphism between smooth projective varieties of dimension n. Let L be a line bundle on Y such that $\Omega_Y(L)$ is globally generated. Then*

$$\deg(f) \cdot c_n(\Omega_Y(L)) \;\leq\; c_n(\Omega_X(f^*L)).$$

Corollary 2.2. *Let X and Y be smooth subvarieties of $\mathbb{P}^{n+2}$ of dimension $n \geq 3$ cut out by two smooth hypersurfaces of degree x_1, x_2 and y_1, y_2 respectively. If there is a surjective morphism $f : X \to Y$, then*

$$\frac{1}{y_1 y_2} c_n(\Omega_Y(2)) \;\leq\; \frac{1}{m^n \cdot x_1 x_2} c_n(\Omega_X(2m)), \tag{2.1}$$

where m is the number given by $f^*\mathcal{O}_Y(1) \cong \mathcal{O}_X(m)$ for the ample generators $\mathcal{O}_X(1)$ and $\mathcal{O}_Y(1)$ of X and Y respectively.

Proof. Since $\Omega(2)$ is globally generated on $\mathbb{P}^{n+2}$, so is its quotient $\Omega_Y(2)$. By Lemma 2.1,

$$\deg(f) \cdot c_n(\Omega_Y(2)) \ \leq \ c_n(\Omega_X(2m)).$$

The inequality (2.1) follows from $\deg(f) = m^n \dfrac{\mathcal{O}_X(1)^n}{\mathcal{O}_Y(1)^n} = m^n \dfrac{x_1 x_2}{y_1 y_2}$. $\square$

Now we compute the involved Chern numbers explicitly. Let Z be a smooth subvariety of $\mathbb{P}^{n+2}$ cut out by two smooth hypersurfaces H_1 and H_2 of degree d_1 and d_2 respectively. Assume that $n \geq 3$ so that we get the isomorphism $\mathrm{Pic}(Z) \cong \mathrm{Pic}(\mathbb{P}^{n+2}) \cong \mathbb{Z}$ given by the restriction of line bundles.

We may use the (twisted) Euler sequence and the conormal sequences:

$$\begin{aligned}
0 \to \quad & \Omega_{\mathbb{P}^{n+2}}(2m)|_Z \to \mathcal{O}_Z(2m-1)^{\oplus(n+3)} \to \mathcal{O}_Z(2m) \to 0, \\
0 \to \quad & \mathcal{O}_{H_1}(-d_1) \to \Omega_{\mathbb{P}^{n+2}}|_{H_1} \to \Omega_{H_1} \to 0, \\
0 \to \quad & \mathcal{O}_Z(-d_2) \to \Omega_{H_1}|_Z \to \Omega_Z \to 0.
\end{aligned}$$

From these we get

$$\begin{aligned}
& c(\Omega_Z(2m)) \\
& = (1+(2m-1)h)^{n+3}(1+2mh)^{-1}(1+(2m-d_1)h)^{-1}(1+(2m-d_2)h)^{-1},
\end{aligned}$$

where h is the hyperplane section class of Z. Therefore the top Chern class is computed by the residue at 0:

$$c_n(\Omega_Z(2m)) = h^n \cdot \mathbf{Res}_0\ (\omega),$$

where

$$\omega = \frac{(1+(2m-1)z)^{n+3}}{z^{n+1}(1+2mz)(1+(2m-d_1)z)(1+(2m-d_2)z)} dz.$$

Now we can use the residue theorem to compute $c_n(\Omega_Z(2m))$.
(Case 1) If $d_1, d_2, 2m$ are different to each other, then

$$\begin{aligned}
\mathbf{Res}_0\, \omega = -\, (& \mathbf{Res}_{-1/2m}\, \omega \ + \ \mathbf{Res}_{-1/(2m-d_1)}\, \omega \\
& + \ \mathbf{Res}_{-1/(2m-d_2)}\, \omega \ + \ \mathbf{Res}_{\infty}\, \omega),
\end{aligned}$$

where

$$\begin{aligned}
\mathbf{Res}_{-1/2m}\,\omega &= \frac{(-1)^{n+1}}{2md_1d_2},\\
\mathbf{Res}_{-1/(2m-d_i)}\,\omega &= \frac{(d_i-1)^{n+3}}{d_i(2m-d_i)(d_i-d_j)}, \quad 1\le i\ne j\le 2,\\
\mathbf{Res}_{\infty}\,\omega &= -\frac{(2m-1)^{n+3}}{2m(2m-d_1)(2m-d_2)}.
\end{aligned}$$

(Case 2) If $d_1 = d_2 \geq 3$, then

$$\mathbf{Res}_0\,\omega = -(\mathbf{Res}_{-1/2m}\,\omega \;+\; \mathbf{Res}_{1/(d_1-2m)}\,\omega \;+\; \mathbf{Res}_{\infty}\,\omega),$$

where

$$\begin{aligned}
\mathbf{Res}_{-1/2m}\omega &= \frac{(-1)^{n+1}}{2md_1^2},\\
\mathbf{Res}_{-1/(2m-d_1)}\omega &= \frac{(d_1-1)^{n+2}}{d_1^2(d_1-2)^2}[2m\{(n+2)d_1+1\}\\
&\qquad -(n+1)d_1^2-2d_1],\\
\mathbf{Res}_{\infty}\omega &= -\frac{(2m-1)^{n+3}}{2m(2m-d_1)^2}.
\end{aligned}$$

(Case 3) If $d_1 \geq 3, d_2 = 2$, and $m = 1$, then

$$\mathbf{Res}_0\,\omega = -(\mathbf{Res}_{-1/2}\,\omega \;+\; \mathbf{Res}_{1/(d_1-2)}\,\omega \;+\; \mathbf{Res}_{\infty}\,\omega),$$

where

$$\begin{aligned}
\mathbf{Res}_{-1/2}\,\omega &= \frac{(-1)^{n+1}}{4d_1},\\
\mathbf{Res}_{1/(d_i-2)}\,\omega &= -\frac{(d_1-1)^{n+3}}{d_1(d_1-2)^2},\\
\mathbf{Res}_{\infty}\,\omega &= \frac{n+3}{2(d_1-2)} - \frac{d_1-4}{4(d_1-2)^2}.
\end{aligned}$$

From these computations, we get the following formulae.

LEMMA 2.3. *(1) If $d_1 = 3, d_2 = 2$ and $m \geq 2$, then*

$$\begin{aligned}
\frac{1}{6}c_n(\Omega_Z(2m)) = &\frac{(2m-1)^{n+3}}{2m(2m-2)(2m-3)} - \frac{2^{n+3}}{3(2m-3)}\\
&+\frac{1}{2(2m-2)} + \frac{(-1)^n}{12m}.
\end{aligned}$$

(2) If $d_1 = d_2 = 3$, then

$$\frac{1}{9}c_n(\Omega_Z(2m)) = \frac{(2m-1)^{n+3}}{2m(2m-3)^2} - \frac{2^{n+2}}{9(2m-3)^2}(6mn+14m-9n-15) + \frac{(-1)^n}{18m}.$$

(3) If $d_1 > d_2 \geq 3$, then

$$\frac{1}{d_1d_2}c_n(\Omega_Z(2)) = \frac{1}{d_1-d_2}\left(\frac{(d_1-1)^{n+3}}{d_1(d_1-2)} - \frac{(d_2-1)^{n+3}}{d_2(d_2-2)}\right) + \frac{1}{2(d_1-2)(d_2-2)} + \frac{(-1)^n}{2d_1d_2}.$$

(4) If $d_1 = d_2 \geq 3$, then

$$\frac{1}{d_1^2}c_n(\Omega_Z(2)) = \frac{n(d_1-1)^{n+2}}{d_1(d_1-2)} + \frac{(d_1-1)^{n+2}(d_1^2-2d_1-2)}{d_1^2(d_1-2)^2} + \frac{1}{2(d_1-2)^2} + \frac{(-1)^n}{2d_1^2}.$$

(5) If $d_1 \geq 3$ and $d_2 = 2$, then

$$\frac{1}{2d_1}c_n(\Omega_Z(2)) = \frac{(d_1-1)^{n+3}}{d_1(d_1-2)^2} + \frac{d_1-4}{4(d_1-2)^2} - \frac{n+3}{2(d_1-2)} + \frac{(-1)^n}{4d_1}.$$

LEMMA 2.4. *(1) If $m \geq 2$, $d_1 = 3$, and either $d_2 = 2$ or $d_2 = 3$, then*

$$\frac{1}{m^n \cdot d_1d_2}c_n(\Omega_Z(2m)) \ < \ 2^{n+1}.$$

(2) If either $d_1, d_2 \geq 3$ or $d_1 > d_2 = 2$, then

$$\frac{1}{d_1d_2}c_n(\Omega_Z(2)) > (d_1-1)^n.$$

Proof. (1) If $m \geq 2$, $d_1 = 3$, and $d_2 = 2$, by Lemma 2.3 (1),

$$\frac{1}{6m^n}c_n(\Omega_Z(2m)) \ < \ \frac{(2m-1)^{n+3}}{m^n \cdot 2m(2m-2)(2m-3)}.$$

The righthand side is smaller than 2^{n+1} for $m = 2$. For $m \geq 3$, still it is bounded by 2^{n+1}, since

$$\frac{(2m-1)^3}{2m(2m-2)(2m-3)} \ < \ 2.$$

If $m \geq 2$, $d_1 = 3$, and $d_2 = 3$, by Lemma 2.3 (2),

$$\frac{1}{9m^n} c_n(\Omega_Z(2m)) \;<\; \frac{(2m-1)^{n+3}}{m^n \cdot 2m(2m-3)^2}.$$

The righthand side is smaller than 2^{n+1} for $m = 2, 3$. If $m \geq 4$, then still it is bounded by 2^{n+1}, since

$$\frac{(2m-1)^3}{2m(2m-3)^2} \;<\; 2.$$

(2) We consider the three cases corresponding to (3), (4), and (5) of Lemma 2.3 in turn. First if $d_1 > d_2 \geq 3$, then the wanted inequality is obtained from the following claim.

For integers a, b such that $a > b \geq 2$ and for any positive integer n,

$$\frac{1}{a-b}\left(\frac{a^{n+3}}{a^2-1} - \frac{b^{n+3}}{b^2-1}\right) \;>\; a^n.$$

This can be easily shown by induction on n.

Secondly if $d_1 = d_2 \geq 3$, then from (4) of Lemma 2.3, we easily get a much stronger inequality than wanted.

Finally if $d_1 \geq 3$ and $d_2 = 2$, then since

$$(d_1-1)^{n+3} \;>\; d_1^2(d_1-2)^2(d_1-1)^{n-1},$$

we get

$$\frac{(d_1-1)^{n+3}}{d_1(d_1-2)^2} \;>\; d(d_1-1)^{n-1} = (d_1-1)^n + (d_1-1)^{n-1}.$$

Since $(d_1 - 1)^{n-1} > \dfrac{n+3}{2(d_1-2)}$, the wanted inequality follows from (5) of Lemma 2.3. $\square$

3. Proof of Theorem 1.6

Let X be the subvariety of $\mathbb{P}^{n+2}$, $\dim X = n \geq 3$, as was described in the statement of Theorem 1.6. Let Y be a smooth subvariety of $\mathbb{P}^{n+2}$ cut out by two smooth hypersurfaces of degree d_1 and d_2, $d_1 \geq d_2$. Assuming that there is a surjective morphism

$$f : X \to Y$$

of degree bigger than 1, we have to show that either $d_1 = d_2 = 2$ or $d_1 d_2 \leq 3$.

Let m be the integer such that $f^*\mathcal{O}_Y(1) \cong \mathcal{O}_X(m)$. First we assume $m \geq 2$ and prove that either $d_1 = d_2 = 2$ or $d_1 d_2 \leq 3$.

By the inequality (2.1),

$$\frac{1}{d}c_n(\Omega_Y(2)) \ \leq \ \frac{1}{dm^n}c_n(\Omega_X(2m)), \tag{3.1}$$

where $d = 6$ (resp. $d = 9$) if X is an intersection of a quadric and a cubic (resp. an intersection of two cubics). By Lemma 2.4 (1), the right-hand side is bounded from above by 2^{n+1}.

If either $d_1, d_2 \geq 3$ or $d_1 > d_2 \geq 2$, then by Lemma 2.4 (2), the left-hand side is bounded from below by $(d_1 - 1)^n$. Hence we get

$$d_1 - 1 < 2^{1/n} \cdot 2,$$

which implies $d_1 \leq 3$. Now we need to exclude the cases $d_1 = d_2 = 3$ and $d_1 = 3, d_2 = 2$. If $d_1 = d_2 = 3$, then by Lemma 2.3 (4),

$$\frac{1}{d_1 d_2}c_n(\Omega_Y(2)) \ > \ \frac{n \cdot 2^{n+2}}{3} \geq 2^{n+2}.$$

If $d_1 = 3$ and $d_2 = 2$, then by Lemma 2.3 (5),

$$\frac{1}{d_1 d_2}c_n(\Omega_Y(2)) \ > \ \frac{2^{n+3}}{3} - n.$$

Both of these contradict to the inequality (3.1) together with the upper bound 2^{n+1} on the righthand side.

Now we finish the proof by considering the case where $m = 1$. In this case, $\deg f = 6/(d_1 d_2)$ (resp. $\deg f = 9/(d_1 d_2)$). If $d_1 d_2 = 6$ (resp. $d_1 d_2 = 9$), then $\deg f = 1$ and f must be an isomorphism. Otherwise, $d_1 d_2 \leq 3$. This finishes the proof of Theorem 1.6. □

Remark 3.1. When $d_1 = 3$ and $d_2 = 1$, we can prove that $m = 1$ by computing the Chern number $c_n(\Omega_Y(2))$ explicitly. On the other hand, the Chern number inequality does not give any effective result for the cases where $d_1, d_2 \leq 2$.

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