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ON MINIMAL PRECONTINUOUS FUNCTIONS

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Abstract. In this paper, we introduce the notions of minimal precontinuity, m-preclosed graph, almost m-precompact and m-precompact and investigate some properties for such notions.

1. Introduction

In [3], Popa and Noiri introduced the notion of minimal structure which is a generalization of a topology on a given nonempty set. And they introduced the notion of m-continuous function as a function defined between a minimal structure and a topological space. They showed that the m-continuous functions have properties similar to those of continuous functions between topological spaces. In [2], we introduced the notions of m-preopen sets defined on minimal structures and investigated some basic properties. In this paper, we introduce and study the notion of m-precontinuous function defined between a minimal structure and a topological space. The notion of m-precontinuous function is a generalization of m-continuous function defined between a minimal structure and a topological space. We also introduce and study the notions of m-preclosed graph, m-precompact and almost m-precompact.

2. Preliminaries

Let X be a topological space and $A \subseteq X$. The closure of A and the interior of A are denoted by $cl(A)$ and $int(A)$, respectively. A subfamily m_X of the power set $P(X)$ of a nonempty set X is called a *minimal structure* [3] on X if $\emptyset \in m_X$ and $X \in m_X$. By (X, m_X), we denote a nonempty set X with a minimal structure m_X on X. Simply we call

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(X, m_X) a space with a minimal structure m_X on X. Elements in m_X are called m-open sets. Let (X, m_X) be a space with a minimal structure m_X on X. For a subset A of X, the m-closure of A and the m-interior of A are defined as the following [3]:

$$mInt(A) = \cup\{U : U \subseteq A, U \in m_X\};$$

$$mCl(A) = \cap\{F : A \subseteq F, X - F \in m_X\}.$$

Definition 2.1. ([2]) Let (X, m_X) be a space with a minimal structure m_X on X and $A \subseteq X$. Then a set A is called an m-preopen set in X if

$$A \subseteq mInt(mCl(A)).$$

A set A is called an *m-preclosed* set if the complement of A is m-preopen.

Definition 2.2. ([2]) Let (X, m_X) be a space with a minimal structure m_X. For $A \subseteq X$, the *m-pre-closure* and the *m-pre-interior* of A, denoted by $mpCl(A)$ and $mpInt(A)$, respectively, are defined as the following:

$$mpCl(A) = \cap\{F \subseteq X : A \subseteq F, F \text{ is } m\text{-preclosed in } X\}$$

$$mpInt(A) = \cup\{U \subseteq X : U \subseteq A, U \text{ is } m\text{-preopen in } X\}.$$

Theorem 2.3. *([2]) Let (X, m_X) be a space with a minimal structure m_X and $A \subseteq X$. Then*
(1) $mpInt(A) \subseteq A$.
(2) If $A \subseteq B$, then $mpInt(A) \subseteq mpInt(B)$.
(3) A is m-preopen iff $mpInt(A) = A$.
(4) $mpInt(mpInt(A)) = mpInt(A)$.
(5) $mpCl(X - A) = X - mpInt(A)$ and $mpInt(X - A) = X - mpCl(A)$.

Theorem 2.4. *([2]) Let (X, m_X) be a space with a minimal structure m_X and $A \subseteq X$. Then*
(1) $A \subseteq mpCl(A)$.
(2) If $A \subseteq B$, then $mpCl(A) \subseteq mpCl(B)$.
(3) F is m-preclosed iff $mpCl(F) = F$.
(4) $mpCl(mpCl(A)) = mpCl(A)$.

Let $f : (X, m_X) \to (Y, \tau)$ be a function between a space (X, m_X) with minimal structure m_X and a topological space (Y, τ). Then f is said to be *m-continuous* [3] if for each x and each open set V containing $f(x)$, there exists an m-open set U containing x such that $f(U) \subseteq V$.

3. Minimal precontinuous functions

Definition 3.1. Let $f : (X, m_X) \to (Y, \tau)$ be a function between a space X with a minimal structure m_X and a topological space Y. Then f is said to be *minimal precontinuous* (briefly *m-precontinuous*) if for each x and each open set V containing $f(x)$, there exists an m-preopen set U containing x such that $f(U) \subseteq V$.

$$m\text{-continuity} \Rightarrow m\text{-precontinuity}$$

In the above diagram, the converse may not be true.

Example 3.2. *Let $X = \{a, b, c\}$. Consider a minimal structure $m_X = \{\emptyset, \{a\}, \{b\}, X\}$ and a topology $\tau = \{\emptyset, \{a\}, \{a, b\}, X\}$. Then the identity function $f : (X, m_X) \to (X, \tau)$ is m-precontinuous but not m-continuous.*

Theorem 3.3. *Let $f : (X, m_X) \to (Y, \tau)$ be a function between a space X with a minimal structure m_X and a topological space (Y, τ). Then the following statements are equivalent:*
(1) f is m-precontinuous.
(2) For each open set V in Y, $f^{-1}(V)$ is m-preopen.
(3) For each closed set B in Y, $f^{-1}(B)$ is m-preclosed.
(4) $f(mpCl(A)) \subseteq cl(f(A))$ for $A \subseteq X$.
(5) $mpCl(f^{-1}(B)) \subseteq f^{-1}(cl(B))$ for $B \subseteq Y$.
(6) $f^{-1}(int(B)) \subseteq mpInt(f^{-1}(B))$ for $B \subseteq Y$.

Proof. (1) $\Rightarrow$ (2) For any open set V in Y and for each $x \in f^{-1}(V)$. From m-precontinuity of f, there exists an m-preopen set U containing x such that $f(U) \subseteq V$. This implies $x \in U \subseteq f^{-1}(V)$ for each $x \in f^{-1}(V)$. Since any union of m-preopen sets is m-preopen (Theorem 3.4 [2]), $f^{-1}(V)$ is m-preopen.

(2) $\Rightarrow$ (3) Obvious.

(3) $\Rightarrow$ (4) For $A \subseteq X$,

$$\begin{aligned}
&f^{-1}(cl(f(A)))\\
&\quad = f^{-1}(\cap\{F \subseteq Y : f(A) \subseteq F \text{ and } F \text{ is closed}\})\\
&\quad = \cap\{f^{-1}(F) \subseteq X : A \subseteq f^{-1}(F) \text{ and } f^{-1}(F) \text{ is } m\text{-preclosed}\}\\
&\quad \supseteq \cap\{K \subseteq X : A \subseteq K \text{ and } K \text{ is } m\text{-preclosed}\}\\
&\quad = mpCl(A)
\end{aligned}$$

Therefore, $f(mpCl(A)) \subseteq cl(f(A))$.

(4) $\Leftrightarrow$ (5) Obvious.

(5) $\Leftrightarrow$ (6) Obvious.

(6) $\Rightarrow$ (1) For $x \in X$ and for each open set V containing $f(x)$, from (6), it follows $x \in f^{-1}(V) = f^{-1}(int(V)) \subseteq mpInt(f^{-1}(V))$. So there exists an m-preopen set U containing x such that $x \in U \subseteq f^{-1}(V)$, i.e. $f(U) \subseteq V$. Hence f is m-precontinuous. $\square$

LEMMA 3.4. *([2]) Let (X, m_X) be a space with a minimal structure m_X and $A \subseteq X$. Then*
(1) $mCl(mInt(A)) \subseteq mCl(mInt(mpCl(A))) \subseteq mpCl(A)$.
(2) $mpInt(A) \subseteq mInt(mCl(mpInt(A))) \subseteq mInt(mCl(A))$.

From Theorem 3.3 and Lemma 3.4, obviously the next theorem is obtained.

THEOREM 3.5. *Let $f : (X, m_X) \to (Y, \tau)$ be a function between a space X with a minimal structure m_X and a topological space (Y, τ). Then the following statements are equivalent:*
(1) f is m-precontinuous.
(2) $f^{-1}(V) \subseteq mInt(mCl(f^{-1}(V)))$ for each open set V in Y.
(3) $mCl(mInt(f^{-1}(F))) \subseteq f^{-1}(F)$ for each closed set F in Y.
(4) $f(mCl(mInt(A))) \subseteq cl(f(A))$ for $A \subseteq X$.
(5) $mCl(mInt(f^{-1}(B))) \subseteq f^{-1}(cl(B))$ for $B \subseteq Y$.
(6) $f^{-1}(int(B)) \subseteq mInt(mCl(f^{-1}(B)))$ for $B \subseteq Y$.

DEFINITION 3.6. Let $f : (X, m_X) \to (Y, \tau)$ be a function between a space (X, m_X) with a minimal structure m_X and a topological space (Y, τ). Then f has an *m-preclosed graph* if for each $(x, y) \in (X \times Y) - G(f)$, there exist an m-preopen set U containing x and an open set V containing y such that $(U \times V) \cap G(f) = \emptyset$.

LEMMA 3.7. *Let $f : (X, m_X) \to (Y, \tau)$ be a function between a space (X, m_X) with a minimal structure m_X and a topological space (Y, τ). Then f has an m-preclosed graph if and only if for each $(x, y) \in (X \times Y) - G(f)$, there exist an m-preopen set U containing x and an open set V containing y such that $f(U) \cap V = \emptyset$.*

THEOREM 3.8. *Let $f : (X, m_X) \to (Y, \tau)$ be a function between a space (X, m_X) with a minimal structure m_X and a topological space (Y, τ). If f is m-precontinuous and (Y, τ) is T_2, then $G(f)$ is an m-preclosed graph.*

Proof. Let $(x, y) \in (X \times Y) - G(f)$; then $f(x) \neq y$. Since Y is T_2, there are disjoint open sets U, V such that $f(x) \in U$, $y \in V$. Then for $f(x) \in U$, by m-precontinuity, there exists an m-preopen set G containing x such that $f(G) \subseteq U$. Consequently, there exist an open set V and m-preopen set G containing y, x, respectively, such that $f(G) \cap V = \emptyset$. Therefore, by Lemma 3.7, $G(f)$ is m-preclosed. □

Definition 3.9. Let (X, m_X) be a space with a minimal structure m_X. Then X is said to be *m-pre-T_2* if for any distinct points x and y of X, there exist disjoint m-preopen sets U, V such that $x \in U$ and $y \in V$.

Theorem 3.10. *Let $f : (X, m_X) \to (Y, \tau)$ be a function between a space (X, m_X) with a minimal structure m_X and a topological space (Y, τ). If f is an injective and m-precontinuous function and if Y is T_2, then X is m-pre-T_2.*

Proof. Obvious. □

Theorem 3.11. *Let $f : (X, m_X) \to (Y, \tau)$ be a function between a space (X, m_X) with a minimal structure m_X and a topological space (Y, τ). If f is an injective m-precontinuous function with an m-preclosed graph, then X is m-pre-T_2.*

Proof. Let x_1 and x_2 be any distinct points of X. Then $f(x_1) \neq f(x_2)$, so $(x_1, f(x_2)) \in (X \times Y) - G(f)$. Since the graph $G(f)$ is m-preclosed, there exist an m-preopen set U containing x_1 and $V \in \tau$ containing $f(x_2)$ such that $f(U) \cap V = \emptyset$. Since f is m-precontinuous, $f^{-1}(V)$ is an m-preopen set containing x_2 such that $U \cap f^{-1}(V) = \emptyset$. Hence X is m-pre-T_2. □

Definition 3.12. A subset A of a space (X, m_X) with a minimal structure m_X is said to be *m-precompact* (resp. *almost m-precompact*) relative to A if every collection $\{U_i : i \in J\}$ of m-preopen subsets of X such that $A \subseteq \cup\{U_i : i \in J\}$, there exists a finite subset J_0 of J such that $A \subseteq \cup\{U_j : j \in J_0\}$ (resp. $A \subseteq \cup\{mpCl(U_j) : j \in J_0\}$). A subset A of a minimal structure (X, m_X) is said to be *m-precompact* (resp. *almost m-precompact*) if A is m-precompact (resp. almost m- precompact) as a subspace of X.

Theorem 3.13. *Let $f : (X, m_X) \to (Y, \tau)$ be an m-precontinuous function between a space (X, m_X) with a minimal structure m_X and a topological space (Y, τ). If A is an m-precompact set, then $f(A)$ is compact.*

Proof. Obvious. □

We recall that a topological space (X, τ) is said to be *quasi H-closed* [5] if for each open cover $\mathbf{C} = \{G_i \subseteq X : G_i \text{ is open}, i \in I\}$, there exists a finite index set $F \subseteq I$ such that $X = \cup_{i \in F} cl(G_i)$.

THEOREM 3.14. *Let $f : (X, m_X) \to (Y, \tau)$ be an m-precontinuous function between a space (X, m_X) with a minimal structure m_X and a topological space (Y, τ). If A is a almost m-precompact set, then $f(A)$ is quasi-H-closed.*

Proof. Let $\{U_i : i \in J\}$ be an open cover of $f(A)$ in Y. Then since f is an m-precontinuous function, $\{f^{-1}(U_i) : i \in J\}$ is an m-preopen cover of A in X. By m-precompactness, there exists $J_0 = \{j_1, j_2, \cdots, j_n\} \subseteq J$ such that $A \subseteq \cup_{j \in J_0} mpCl(f^{-1}(U_j))$. From Theorem 3.3 (5), it implies $f(A) \subseteq f(\cup_{j \in J_0} mpCl(f^{-1}(U_j))) \subseteq f(\cup_{j \in J_0} f^{-1}(cl(U_j))) \subseteq \cup_{j \in J_0} cl(U_j)$. Thus $f(A)$ is quasi-H-closed. □

DEFINITION 3.15. Let $f : (X, \tau) \to (Y, m_Y)$ be a function between a topological space (X, τ) and a space (Y, m_Y) with a minimal structure m_Y. Then f is said to be *m-preopen* if for each open set U in X, $f(U)$ is m-preopen.

THEOREM 3.16. *Let $f : (X, \tau) \to (Y, m_Y)$ be a function between a topological space (X, τ) and a space (Y, m_Y) with a minimal structure m_Y. Then the following statements are equivalent:*
(1) f is m-preopen.
(2) $f(int(A)) \subseteq mpInt(f(A))$ for $A \subseteq X$.
(3) $int(f^{-1}(B)) \subseteq f^{-1}(mpInt(B))$ for $B \subseteq Y$.

Proof. (1) ⇒ (2) For $A \subseteq X$, by (1) and Theorem 2.3 (3),

$$f(int(A)) = mpInt(f(int(A))) \subseteq mpInt(f(A)).$$

Hence $f(int(A)) \subseteq mpInt(f(A))$.

(2) ⇒ (3) For $B \subseteq Y$,

$$f(int(f^{-1}(B))) \subseteq mpInt(f(f^{-1}(B))) \subseteq mpInt(B)$$

This implies $int(f^{-1}(B)) \subseteq f^{-1}(mpInt(B))$.

(3) ⇒ (1) For each open set U in X, from $int(f(U)) \subseteq mpInt(f(U))$, we have $U = int(U) \subseteq int(f^{-1}(f(U))) \subseteq f^{-1}(mpInt(f(U)))$. This implies $f(U) \subseteq mpInt(f(U))$, and from Theorem 2.3 (3), $f(U)$ is m-preopen. □

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