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COMMON FIXED POINT THEOREMS FOR GENERALIZED CONTRACTIVE TYPE MULTIVALUED MAPPINGS

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Abstract. In this paper, we give a generalized contractive type condition for a pair of multivalued mappings and analyze the existence of common fixed point for these mappings.

1. Introduction

In [2,5], the authors proved some fixed point theorems for self maps of a metric space satisfying integal type contractive condition. In [6], the author gave a generalized contractive type condition for a pair of self maps of a metric space and proved some common fixed point theorems for these maps.

In this paper, we have a generalization of the results of [6] to multivalued maps. And, we give a generalized contractive type condition for a pair of multivalued maps and prove some common fixed point theorems for these maps. From the our main results, we have some fixed point theorems for multivalued maps satisfying integal type contractive condition.

Let (X, d) be a metric space. We denote by $CB(X)$ the family of nonempty closed bounded subsets of X. Let $H(\cdot, \cdot)$ be the Hausdorff distance on $CB(X)$, i.e.,

$$H(A, B) = max\{sup_{a\in A}d(a, B), sup_{b\in B}d(b, A)\}, \quad A, B \in CB(X),$$

where $d(a, B) = inf\{d(a, b) : b \in B\}$ is the distance from the point a to the subset B.

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For $A, B \in CB(X)$, let $D(A,B) = sup_{x\in A} inf_{y\in B} d(x,y)$. Then we have $D(A,B) \le H(A,B)$ for all $A, B \in CB(X)$.

From now on, we denote

$$M(x,y) = max\{d(x,y), d(x,Sx), d(y,Ty), \frac{1}{2}\{d(x,Ty) + d(y,Sx)\}\}$$

for multivalued maps $S, T : X \to CB(X)$ and $x, y \in X$.

For $a \in (0,\infty)$, let $\Im[0,a)$ be the class of strictly increasing continuous function $F : [0,a) \to [0,\infty)$ satisying $F(0) = 0$.

Note that if $F(s) = \int_0^s \varphi(t)dt$, then $F \in \Im[0,a)$ where $\varphi : [0,a) \to [0,\infty)$ is a Lebesque measurable function such that $\varphi > 0$ almost everywhere and $\int_0^b \varphi(t)dt < \infty$ for each $b \in (0,a)$.

For $F \in \Im[0,a)$, we denote $\Psi[0, F(a-0))$ the class of strictly increasing right upper semi-continuous function $\phi : [0, F(a-0)) \to [0,\infty)$ satisfying
($\phi1$) $0 < \phi(t) < t$ for all $t \in (0, F(a-0))$,
($\phi2$) for each $t \in (0,a)$, $\sum_{n=1}^{\infty} F^{-1}(\phi^n(F(t))) < \infty$.

Note that $\phi(0) = 0$ and $\lim_{n\to\infty} F^{-1}(\phi^n(F(t))) = 0$ for each $t \in (0,a)$.

2. Main Theorem

In this section, we consider some generalized contractive type common fixed point theorems for a pair of multivalued maps.

For a metric space (X,d), let $V = sup\{H(A,B) : A, B \in CB(X)\}$. Let $a = V$ if $V = \infty$ and $a > V$ if $V < \infty$.

THEOREM 2.1. *Let (X,d) be a complete metric space. Suppose that $T, S : X \to CB(X)$ are multivalued maps, $F \in \Im[0,a)$ and $\phi \in \Psi[0, F(a-0))$ satisfying for each $x, y \in X$,*

$$F(H(Sx,Ty)) \le \phi(F(M(x,y))). \tag{2.1}$$

Then T and S have a common fixed point in X. That is, there exists a point $p \in X$ such that $p \in Tp \cap Sp$.

Proof. Let $x_0 \in X$ and let c be a real number with $d(x_0, Sx_0) < c$. The we can take $x_1 \in Sx_0$ such that $d(x_0, x_1) < c$.

We have

$$\begin{aligned}
&F(d(x_1, Tx_1))\\
\leq &F(H(Sx_0, Tx_1))\\
\leq &\phi(F(\max\{d(x_0, x_1), d(x_0, Sx_0), d(x_1, Tx_1),\\
&\quad \frac{1}{2}\{d(x_0, Tx_1) + d(x_1, Sx_0)\}\}))\\
\leq &\phi(F(\max\{d(x_0, x_1), d(x_0, x_1), d(x_1, Tx_1),\\
&\quad \frac{1}{2}\{d(x_0, Tx_1) + d(x_1, x_1)\}\}))\\
\leq &\phi(F(d(x_0, x_1)))\\
< &\phi(F(c)),
\end{aligned}$$

because F and ϕ are strictly increasing. Thus we have $d(x_1, Tx_1) < F^{-1}(\phi(F(c)))$. We can take $x_2 \in Tx_1$ such that $d(x_1, x_2) < F^{-1}(\phi(F(c)))$.

Similarly we have

$$\begin{aligned}
&F(d(x_2, Sx_2))\\
\leq &F(H(Sx_2, Tx_1))\\
\leq &\phi(F(\max\{d(x_1, x_2), d(x_2, Sx_2), d(x_1, Tx_1),\\
&\quad \frac{1}{2}\{d(x_2, Tx_1) + d(x_1, Sx_2)\}\}))\\
\leq &\phi(F(\max\{d(x_1, x_2), d(x_2, Sx_2), d(x_1, x_2),\\
&\quad \frac{1}{2}\{d(x_2, x_2) + d(x_1, Sx_2)\}\}))\\
\leq &\phi(F(d(x_1, x_2))).
\end{aligned}$$

Thus we have

$$d(x_2, Sx_2) \leq F^{-1}(\phi(F(d(x_1, x_2)))) < F^{-1}(\phi^2(F(c)))).$$

We can take $x_3 \in Sx_2$ such that $d(x_2, x_3) < F^{-1}(\phi^2(F(c)))$.

Continuing this process, we can find a sequence $\{x_n\}$ in X such that $x_{2n+1} \in Sx_{2n}, x_{2n+2} \in Tx_{2n+1}, d(x_n, x_{n+1}) < F^{-1}(\phi^n(F(c)))$.

Thus we have $\sum_{n=1}^{\infty} d(x_n, x_{n+1}) < \sum_{n=1}^{\infty} F^{-1}(\phi^n(F(c))) < \infty$.

Therefore, $\{x_n\}$ is a Cauchy sequence. By the completeness of X, there exists a $p \in X$ such that $lim_{n\to\infty} x_n = p$.

From (2.1) we have

$$\begin{aligned}
&F(d(x_{2n+1}, Tp)) \leq F(H(Sx_{2n}, Tp)) \\
\leq&\phi(F(\max\{d(x_{2n}, p), d(x_{2n}, Sx_{2n}), d(p, Tp), \\
&\frac{1}{2}\{d(x_{2n}, Tp) + d(p, Sx_{2n})\}\})) \\
\leq&\phi(F(\max\{d(x_{2n}, p), d(x_{2n}, x_{2n+1}), d(p, Tp), \\
&\frac{1}{2}\{d(x_{2n}, Tp) + d(p, x_{2n+1})\}\})).
\end{aligned} \tag{2.2}$$

Letting $n \to \infty$ in (2.2) we have $F(d(p, Tp)) \leq \phi(F(d(p, Tp)))$, which implies $F(d(p, Tp)) = 0$. Thus $d(p, Tp) = 0$ or $p \in Tp$.

Similarly, we can show $p \in Sp$. Therefore, we have $p \in Tp \cap Sp$. $\square$

Corollary 2.2. *Let (X, d) be a complete metric space. Suppose that $T, S : X \to CB(X)$ are multivalued maps, $F \in \Im[0, a)$ and $\phi \in \Psi[0, F(a-0))$ satisfying for each $x, y \in X$, $u \in Sx$ and $v \in Ty$,*

$$F(max\{d(u, Ty), d(v, Sx)\}) \leq \phi(F(M(x, y))).$$

Then T and S have a common fixed point in X.

Corollary 2.3. *Let (X, d) be a complete metric space. Suppose that $T, S : X \to CB(X)$ are multivalued maps, $F \in \Im[0, a)$ and $\phi \in \Psi[0, F(a-0))$ satisfying for each $x, y \in X$,*

$$F(max\{D(Sx, Ty), D(Tx, Sy)\}) \leq \phi(F(M(x, y))).$$

Then T and S have a common fixed point in X.

Corollary 2.4. *Let (X, d) be a complete metric space. Suppose that $T : X \to CB(X)$ is multivalued map, $F \in \Im[0, a)$ and $\phi \in \Psi[0, F(a-0))$ satisfying for each $x, y \in X$,*

$$\begin{aligned}
&F(H(Tx, Ty)) \\
&\leq \phi(F(\max\{d(x, y), d(x, Tx), d(y, Ty), \frac{1}{2}\{d(x, Ty) + d(y, Tx)\}\}))
\end{aligned}$$

Then T has a fixed point in X.

In Theorem 2.1(Corollary 2.4), let $F(s) = s$ for $s \in [0, a)$ and $\phi(t) = kt$ for some $k \in [0, 1)$ and all $t \in [0, F(a - 0))$. Then we have Theorem 3.1[1](Corollary 2.2[1] and Theorem 3.3[3]).

In Theorem 2.1(Corollary 2.4) if $F(s) = \int_0^s \varphi(t)dt$, then we have the next two corollaries, where $\varphi : [0, a) \to [0, \infty)$ is a Lebesque measurable function such that $\varphi > 0$ almost everywhere and $\int_0^b \varphi(t)dt < \infty$ for each $b \in (0, a)$.

Corollary 2.5. *Let (X, d) be a complete metric space. Suppose that $T, S : X \to CB(X)$ are multivalued maps and $\phi \in \Psi[0, \int_0^a \varphi(t)dt)$ satisfying for each $x, y \in X$,*

$$\int_0^{H(Tx,Sy)} \varphi(t)dt \le \phi(\int_0^{M(x,y)} \varphi(t)dt).$$

Then T and S have a common fixed point in X.

Corollary 2.6. *Let (X, d) be a complete metric space. Suppose that $T : X \to CB(X)$ is multivalued map and $\phi \in \Psi[0, \int_0^a \varphi(t)dt)$ satisfying for each $x, y \in X$,*

$$\begin{aligned} &\int_0^{H(Tx,Ty)} \varphi(t)dt \\ &\le \phi(\int_0^{\max\{d(x,y),d(x,Tx),d(y,Ty),\frac{1}{2}\{d(x,Ty)+d(y,Tx)\}\}} \varphi(t)dt). \end{aligned}$$

Then T has a fixed point in X.

We now give an example which satisfies the conditions in Theorem 2.1 but does not satisfy the general contractive condition.

Example 2.7. Let $X = \{\frac{1}{n^2} : n = 1, 2, \cdots\} \cup \{0\}$ with the Euclidean metric d. It is easy to see that (X, d) is a complete metric space and $V = 1$. Let $a = 2$. Let $F(t) = \begin{cases} (\frac{1}{2})^{\frac{1}{\sqrt{t}}} & (0 < t < 2), \\ 0 & (t = 0) \end{cases}$ and $\phi(t) = \frac{1}{2}t$ for $0 \le t < F(a - 0)$. Then $F \in \Im[0, 2)$, $\phi \in \Psi[0, F(a - 0))$ and $F^{-1}(t) = (\frac{1}{log_2 t})^2$. For any $c \in (0,2)$, we have

$$\sum_{n=0}^{\infty} F^{-1}(\phi^n(F(c)) = \sum_{n=0}^{\infty} \frac{1}{(log_2 2^{-n}F(c))^2} = \sum_{n=0}^{\infty} \frac{1}{(n+\frac{1}{\sqrt{c}})^2} < \infty.$$

Let $Tx = Sx = \begin{cases} \{\frac{1}{(n+1)^2}\} & (x = \frac{1}{n^2}, n = 1,2,3,\cdots), \\ \{0\} & (x = 0). \end{cases}$

We now show that (2.1) is satisfied. We consider three cases.

Case 1. Let $x = y$. Then we have $d(x,y) = 0$ and $d(Sx,Ty) = 0$. Thus $F(d(Sx,Ty)) \leq \phi(F(d(x,y)))$. Hence (2.1) is satisfied.

Case 2. Let $x = 0$ and $y = \frac{1}{n^2}$(or $x = \frac{1}{n^2}$ and $y = 0$). Then we have

$$\begin{aligned} & F(H(Sx,Ty)) \\ =& F(H(0, \frac{1}{(n+1)^2}) \\ =& F(\frac{1}{(n+1)^2}) \\ =& \frac{1}{2^{n+1}} = \frac{1}{2} \cdot \frac{1}{2^n} \\ =& \phi(F(d(x,y))) \leq \phi(F(M(x,y))). \end{aligned}$$

Case 3. Let $x = \frac{1}{n^2}$ and $y = \frac{1}{m^2}(m > n)$. Then we have

$$\begin{aligned} & F(H(Sx,Ty)) \\ =& F(H(\frac{1}{(n+1)^2}, \frac{1}{(m+1)^2})) \\ =& F(\frac{1}{(n+1)^2} - \frac{1}{(m+1)^2}) \\ =& F(\frac{(m+1)^2-(n+1)^2}{(n+1)^2(m+1)^2}) \\ =& (\frac{1}{2})^{\frac{(m+1)(n+1)}{\sqrt{m^2-n^2-2n+2m}}} \\ =& (\frac{1}{2})^{\frac{m+1}{\sqrt{m^2-n^2-2n+2m}}} (\frac{1}{2})^{\frac{mn+n}{\sqrt{m^2-n^2-2n+2m}}}. \end{aligned}$$

Since $\frac{m+1}{\sqrt{m^2-n^2-2n+2m}} > 1$ and $\frac{mn+n}{\sqrt{m^2-n^2-2n+2m}} > \frac{mn}{\sqrt{m^2-n^2}}$, we have

$$
\begin{aligned}
&F(H(Sx,Ty)) \\
=&(\frac{1}{2})^{\frac{m+1}{\sqrt{m^2-n^2-2n+2m}}}(\frac{1}{2})^{\frac{mn+n}{\sqrt{m^2-n^2-2n+2m}}} \\
\le&\frac{1}{2}\cdot(\frac{1}{2})^{\frac{mn}{\sqrt{m^2-n^2}}} \\
=&\phi(F(d(x,y)))\le\phi(F(M(x,y))).
\end{aligned}
$$

Thus T and S satisfy all conditions in Theorem 2.1 and $0 \in T0 \cap S0$. If there exists $k \in [0,1)$ such that for any $x, y \in X$

$$
\begin{aligned}
&H(Sx,Ty) \\
&\le k\ max\{d(x,y), d(x,Sx), d(y,Ty), \frac{1}{2}\{d(x,Ty)+d(y,Sx)\}\},
\end{aligned}
$$

then we have, for $x = 0$ and $y = \frac{1}{n^2}$ for $n = 1, 2, 3, \cdots$,

$$
\begin{aligned}
&\frac{1}{(n+1)^2} = H(Sx,Ty) \\
&\le k\ max\{d(x,y), d(x,Sx), d(y,Ty), \frac{1}{2}\{d(x,Ty)+d(y,Sx)\}\} \\
&= k\ max\{\frac{1}{n^2}, 0, \frac{1}{n^2}-\frac{1}{(n+1)^2}, \frac{1}{2}\{\frac{1}{(n+1)^2}+\frac{1}{(n+1)^2}\}\} \\
&= k\frac{1}{n^2},
\end{aligned}
$$

which implies $k \ge (\frac{n}{n+1})^2$ for $n = 1, 2, 3, \cdots$. From this inequality, we have that $k \ge 1$. But it is not possible. Thus S and T does not satisfy the general contractive condition.

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