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DIRECT SUM ON WFI-ALGEBRAS

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Abstract. The notion of subdirect sum and direct sum in WFI-algebras is introduced, and several properties are investigated.

1. Introduction

In 1990, W. M. Wu [8] introduced the notion of fuzzy implication algebras (FI-algebra, for short), and investigated several properties. In [7], Z. Li and C. Zheng introduced the notion of distributive (resp. regular, commutative) FI-algebras, and investigated the relations between such FI-algebras and MV-algebras. In [1], Y. B. Jun discussed several aspects of WFI-algebras, and gave a characterization of a WFI-algebra. He introduced the notion of associative (resp. normal, medial) WFI-algebras, and investigated several properties. He gave conditions for a WFI-algebra to be associative/medial, and provided characterizations of associative/medial WFI-algebras, and showed that every associative WFI-algebra is a group in which every element is an involution. He also verified that the class of all medial WFI-algebras is a variety. Y. B. Jun and S. Z. Song [6] introduced the notions of simulative and/or mutant WFI-algebras and investigated some properties. They established characterizations of a simulative WFI-algebra, and gave a relation between an associative WFI-algebra and a simulative WFI-algebra. They also found some types for a simulative WFI-algebra to be mutant. Jun, Park and Roh [5] introduced the concept of ideals of WFI-algebras. They gave relations between a filter and an ideal, and provided characterizations of an ideal. Also they established an extension property for an ideal. In [2] and [3], Y. B. Jun introduced the concept of perfect filters, concrete filters, mote, beam and osculatory in WFI-algebra. He gave relations

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between various these notions. Y. B. Jun and C. H. Park [4] discussed uncanny filters, and investigated related properties. In this paper, we introduce the notion of subdirect sum and direct sum on WFI-algebra. Also we provide related properties.

2. Preliminaries

Let $K(\tau)$ be the class of all algebras of type $\tau = (2, 0)$. By a *WFI-algebra* we mean a system $\mathfrak{X} = (X, \ominus, 1) \in K(\tau)$ such that for all $x, y, z \in X$:

(a1) $x \ominus (y \ominus z) = y \ominus (x \ominus z)$,
(a2) $(x \ominus y) \ominus \big((y \ominus z) \ominus (x \ominus z)\big) = 1$,
(a3) $x \ominus x = 1$,
(a4) $x \ominus y = y \ominus x = 1 \Rightarrow x = y$.

For the convenience of notation, we shall write $[x, y_1, y_2, \cdots, y_n]$ for

$$(\cdots((x \ominus y_1) \ominus y_2) \ominus \cdots) \ominus y_n.$$

We define $[x, y]^0 = x$, and for $n > 0$, $[x, y]^n = [x, y, y, \cdots, y]$, where y occurs n-times.

LEMMA 2.1. [1] *In a WFI-algebra $\mathfrak{X}$, the following are true:*

(b1) $x \ominus [x, y]^2 = 1$,
(b2) $1 \ominus x = 1 \Rightarrow x = 1$,
(b3) $1 \ominus x = x$,
(b4) $x \ominus y = 1 \Rightarrow (y \ominus z) \ominus (x \ominus z) = 1$, $(z \ominus x) \ominus (z \ominus y) = 1$,
(b5) $(x \ominus y) \ominus 1 = (x \ominus 1) \ominus (y \ominus 1)$,
(b6) $[x, y]^3 = x \ominus y$.

A nonempty subset S of a WFI-algebra $\mathfrak{X}$ is called a *subalgebra* of $\mathfrak{X}$ if $x \ominus y \in S$ whenever $x, y \in S$. A nonempty subset F of a WFI-algebra $\mathfrak{X}$ is called a *filter* of $\mathfrak{X}$ if it satisfies:

(c1) $1 \in F$,
(c2) $x \ominus y \in F$ and $x \in F$ imply $y \in F$ for all $x, y \in X$.

A filter F of a WFI-algebra $\mathfrak{X}$ is said to be *closed* [1] if F is also a subalgebra of $\mathfrak{X}$.

LEMMA 2.2. [1] *Let F be a filter of a WFI-algebra $\mathfrak{X}$. Then F is closed if and only if $x \ominus 1 \in F$ for all $x \in F$.*

LEMMA 2.3. [1] *In a finite WFI-algebra, every filter is closed.*

We now define a relation "$\preceq$" on $\mathfrak{X}$ by $x \preceq y$ if and only if $x \ominus y = 1$. It is easy to verify that a WFI-algebra is a partially ordered set with respect to $\preceq$. For a WFI-algebra $\mathfrak{X}$, the set

$$\mathcal{S}(\mathfrak{X}) := \{x \in X \mid x \preceq 1\}$$

is called the *simulative part* of $\mathfrak{X}$ ([6]). Note that $\mathcal{S}(\mathfrak{X})$ is a subalgebra of $\mathfrak{X}$.

LEMMA 2.4. [6] *Let $\mathfrak{X}$ be a WFI-algebra. Then $\mathcal{S}(\mathfrak{X})$ is a filter of X.*

The *doubly simulative part* of $\mathfrak{X}$ [5] is defined to be the set

$$\mathcal{DS}(\mathfrak{X}) := \{x \in X \mid [x, 1]^2 = x\}.$$

Obviously, $1 \in \mathcal{DS}(\mathfrak{X})$ and $\mathcal{DS}(\mathfrak{X}) \cap \mathcal{S}(\mathfrak{X}) = \{1\}$.

3. Main results

In what follows let $\mathfrak{X}$ denote a WFI-algebra $(X; \ominus, 1)$ unless otherwise specified.

LEMMA 3.1. *For any $\mathfrak{X}$, if $a \in X$, then the following conditions are equivalent:*

(1) $a \leq x \Rightarrow a = x$ *for any* $x \in X$.
(2) $[a, 1]^2 = a$.
(3) *there is* $x \in X$ *such that* $a = x \ominus 1$.

Proof. (1) $\Rightarrow$ (2). By (b1), we have $[a, 1]^2 = a$.
(2) $\Rightarrow$ (1). Let $a \preceq x$ for any $x \in X$. Then we have

$$x \ominus a = x \ominus [a, 1]^2 = (a \ominus 1) \ominus (x \ominus 1) = 1.$$

and so $a = x$ by (a4).
(2) $\Rightarrow$ (3). We have $a = [a, 1]^2 = x \ominus 1$, where $x := a \ominus 1$.
(3) $\Rightarrow$ (2). Suppose that $a = x \ominus 1$ for some $x \in X$. Then we have

$$[a, 1]^2 \ominus a = [x \ominus 1, 1]^2 \ominus (x \ominus 1) = (x \ominus 1) \ominus (x \ominus 1) = 1,$$

and so $[a, 1]^2 = a$ by (a4). □

LEMMA 3.2. [1] *Let $\mathfrak{X}$ be a WFI-algebra. Then the following conditions are equivalent:*

(1) $[a, 1]^2 = a$ *for any* $a \in X$.
(2) $[a, x]^2 = a$ *for any* $a, x \in X$.

We now consider the generated filters in WFI-algebras.

DEFINITION 3.3. Let S be a subset of $\mathfrak{X}$. We call the least filter of $\mathfrak{X}$ containing S, the *generated filter* of $\mathfrak{X}$ by S, denoted by $< S >$.

It is obvious that the intersection of any filter family of $\mathfrak{X}$ is a filter. So the generated filter is well-defined and we have an obvious assertion as follows.

LEMMA 3.4. *Let $\mathfrak{X}$ be a WFI-algebra and $S, T \subseteq X$. If $S \subseteq T$, then $< S > \subseteq < T >$. In particular, if $S = \emptyset$, then $< S > = \{1\}$.*

We denote $< \{a_1, a_2, \cdots, a_n\} >$ by $< a_1, a_2, \cdots, a_n >$ in brevity. Sometimes, the filter $< a >$ generated by one element a is also called a *principal filter* of $\mathfrak{X}$. The next theorem gives the description of elements in $< S >$.

THEOREM 3.5. *Let S be a nonempty subset of $\mathfrak{X}$ and let*

$$G := \{x \in X \quad | \quad a_1 \ominus (a_2 \ominus (\cdots \ominus (a_{n-1} \ominus (a_n \ominus x)) \cdots)) = 1 \text{ for some } a_1, a_2, \cdots, a_n \in S\}.$$

Then $< S > = G \cup \{1\}$.

Proof. The proof is straightforward. □

THEOREM 3.6. *Let F be a filter of $\mathfrak{X}$. Define a binary operation $\equiv$ on X as follow:*

$$x \equiv y (mod\, F) \Leftrightarrow x \ominus y \in F \text{ and } y \ominus x \in F$$

for any $x, y \in X$. Then $\equiv$ is a congruence on $\mathfrak{X}$.

Proof. The proof is standard. □

THEOREM 3.7. *Let F be a filter of $\mathfrak{X}$ and $\equiv$ be a congruence relation on $\mathfrak{X}$ defined by* Theorem 3.6. *We denote*

$$\equiv_x := \{y \in X | x \equiv y (mod\, F)\} \text{ and } X/\equiv := \{\equiv_x | x \in X\}.$$

Then the quotient algebra $\mathfrak{X}/\equiv := (X/\equiv; \odot, \equiv_1)$ is a WFI-algebra, where the operation $\odot$ on $X/\equiv$ given by $\equiv_x \odot \equiv_y := \equiv_{x \odot y}$.

Proof. The proof is immediate. □

DEFINITION 3.8. Let F_1 and F_2 be filters of $\mathfrak{X}$ such that X$= < F_1 \cup F_2 >$ and $F_1 \cap F_2 = \{1\}$, then $\mathfrak{X}$ is called the *subdirect sum* of F_1 and F_2, denoted by $\mathfrak{X} = F_1 \bar{\oplus} F_2$.

Example 3.9. Let $X := \{1, a, b\}$ be a set with the following Cayley table.

$\ominus$	1	a	b
1	1	a	b
a	1	1	b
b	1	a	1

Then $\mathfrak{X} = (X; \ominus, 1)$ is a WFI-algebra. It is easy to verify that $F_1 = \{1, a\}$ and $F_2 = \{1, b\}$ are filters of X, and $\mathfrak{X} = F_1 \bar{\oplus} F_2$.

Theorem 3.10. *Let F_1 and F_2 be closed filters of $\mathfrak{X}$. If $\mathfrak{X} = F_1 \bar{\oplus} F_2$. Then there are unique $a \in F_1$ and $b \in F_2$ such that $x \equiv a(mod\, F_2)$ and $y \equiv b(mod\, F_1)$.*

Proof. Let us first prove that there is unique $a \in F_1$ such that $x \equiv a(mod\, F_2)$. Let $x \in X$. Since $X =< F_1 \cup F_2 >$, by Theorem 3.5, we know that there exist $b_1, b_2, \cdots, b_n \in F_2$ such that $b_1 \ominus (b_2 \ominus (\cdots \ominus (b_n \ominus x) \cdots)) = 1$. Put $a := b_1 \ominus (b_2 \ominus (\cdots \ominus (b_n \ominus x) \cdots))$, then $a \in F_1$. Thus we have

$$b_1 \ominus (b_2 \ominus (\cdots \ominus (b_n \ominus (a \ominus x)) \cdots)) = a \ominus a = 1,$$

and so $a \ominus x \in F_2$. Moreover, since $x \ominus a = b_1 \ominus (b_2 \ominus (\cdots \ominus (b_n \ominus 1) \cdots))$, by F_2 being a closed filter of $\mathfrak{X}$, it follows $x \ominus a \in F_2$. Therefore we have $x \equiv a(mod\, F_2)$. Let $a, a' \in F_1$ such that $x \equiv a(mod\, F_2)$ and $x \equiv a'(mod\, F_2)$. By the symmetry and transitivity of congruence, we have $a \equiv a'(mod\, F_2)$, and so $a \ominus a' \in F_2$ and $a' \ominus a \in F_2$. Also, since F_1 is a closed filter of $\mathfrak{X}$ and $a, a' \in F_1$, we obtain $a \ominus a' \in F_1$ and $a' \ominus a \in F_1$. Hence we get $a \ominus a' \in F_1 \cap F_2$ and $a' \ominus a \in F_1 \cap F_2$. Since $F_1 \cap F_2 = \{1\}$, we have $a \ominus a' = 1 = a' \ominus a'$. Therefore we get $a = a'$.

Similarly, there is unique $b \in F_2$ such that $x \equiv b(mod\, F_1)$. This completes the proof. $\square$

Theorem 3.11. *Let F be a closed filter of $\mathfrak{X}$. If $\mathcal{S}(\mathfrak{X}) \cap F = \{1\}$, then $F \subseteq \mathcal{S}(\mathfrak{X})^*$. Further, if $\mathfrak{X} = \mathcal{S}(\mathfrak{X}) \bar{\oplus} F$, then $F = \mathcal{S}(\mathfrak{X})^*$, where $\mathcal{S}(\mathfrak{X})^* := \{x \in X | a \ominus x = x$ for any $a \in \mathcal{S}(\mathfrak{X})\}$.*

Proof. Let $\mathcal{S}(\mathfrak{X}) \cap F = \{1\}$ and $x \in F$. Then by (b1) and Lemma 2.4, we have $[a, x]^2 \in A$ for any $a \in \mathcal{S}(\mathfrak{X})$. Since $x \ominus [a, x]^2 = (a \ominus x) \ominus 1 = (a \ominus 1) \ominus (x \ominus 1) = x \ominus (1 \ominus 1) = x \ominus 1$ and F is a closed filter of $\mathfrak{X}$, we have $[a, x]^2 \in F$. Hence we get $[a, x]^2 = 1$. Also, $x \ominus (a \ominus x) = a \ominus (x \ominus x) = 1$. therefore $a \ominus x = x$ and so $x \in \mathcal{S}(\mathfrak{X})^*$, *i.e.*$F \subseteq \mathcal{S}(\mathfrak{X})^*$.

On the other hand, if $\mathfrak{X} = \mathcal{S}(\mathfrak{X}) \bar{\oplus} F$, then $\mathcal{S}(\mathfrak{X}) \cap F = \{1\}$, and so $F \subseteq \mathcal{S}(\mathfrak{X})^*$. For any $x \in \mathcal{S}(\mathfrak{X})^*$, there is $a \in \mathcal{S}(\mathfrak{X})$ such that $x \equiv a(mod\, F)$ by

Theorem 3.10. Thus we get $a \ominus x \in F$. Note that $a \ominus x = x$. Hence we have $\mathcal{S}(\mathfrak{X})^* \subseteq F$. Therefore $F = \mathcal{S}(\mathfrak{X})^*$. This completes the proof. $\square$

DEFINITION 3.12. Let F_1 and F_2 be filters of $\mathfrak{X}$ and $\mathfrak{X} = F_1 \bar{\oplus} F_2$. If for any $a \in F_1$ and $b \in F_2$, there exists $x \in X$ such that $x \equiv a(mod\, F_2)$ and $y \equiv b(mod\, F_1)$, then we say $\mathfrak{X}$ is the *direct sum* of F_1 and F_2, denoted by $\mathfrak{X} = F_1 \oplus F_2$.

We remark that $\mathcal{DS}(\mathfrak{X})$ is generally not a filter of $\mathfrak{X}$. Let $X := \{1, a, b\}$ be a set with the following Cayley table.

$\ominus$	1	a	b
1	1	a	b
a	1	1	b
b	b	b	1

Then $\mathfrak{X} = (X; \ominus, 1)$ is a WFI-algebra. Then $\mathcal{DS}(\mathfrak{X}) = \{1, b\}$ is not a filter of $\mathfrak{X}$ since $b \ominus a \in \mathcal{DS}(\mathfrak{X}), b \in \mathcal{DS}(\mathfrak{X})$ and $a \notin \mathcal{DS}(\mathfrak{X})$.

THEOREM 3.13. *If $\mathcal{DS}(\mathfrak{X})$ is a filter of $\mathfrak{X}$, then $X = \mathcal{S}(\mathfrak{X}) \oplus \mathcal{DS}(\mathfrak{X})$.*

Proof. For any $x \in X$, let $a \in B$ with $x \preceq a$. Then we have $a \ominus x \in \mathcal{S}(\mathfrak{X})$. Thus $x \in < \mathcal{S}(\mathfrak{X}) \cup \mathcal{DS}(\mathfrak{X}) >$, and so $x =< \mathcal{S}(\mathfrak{X}) \cup \mathcal{DS}(\mathfrak{X}) >$. Therefore we have $\mathfrak{X} = \mathcal{S}(\mathfrak{X}) \bar{\oplus} \mathcal{DS}(\mathfrak{X})$. Also, for any $b \in \mathcal{S}(\mathfrak{X})$ and $p \in \mathcal{DS}(\mathfrak{X})$, putting $x := (p \ominus 1) \ominus b$, we have

$$b \ominus x = (p \ominus 1) \ominus (b \ominus b) = [p, 1]^2 = p \in \mathcal{DS}(\mathfrak{X}),$$

$$x \ominus b = p \ominus 1 \in \mathcal{DS}(\mathfrak{X}).$$

Then $x \equiv b(mod\, \mathcal{DS}(\mathfrak{X}))$. On the other hand, we obtain

$$x \ominus 1 = [p, 1]^2 \ominus (b \ominus 1) = [p, 1]^3 = p \ominus 1.$$

Then we get $p \ominus x \in \mathcal{S}(\mathfrak{X})$ because $b = 1 \ominus b \preceq (p \ominus 1) \ominus (p \ominus b) = p \ominus x, b \in \mathcal{S}(\mathfrak{X})$ and $\mathcal{S}(\mathfrak{X})$ is a filter of X. Moreover, we have

$$x \ominus p = x \ominus [p, 1]^2 = (p \ominus 1) \ominus (x \ominus 1) = 1 \in \mathcal{S}(\mathfrak{X}).$$

So, $x \equiv p(mod\, \mathcal{S}(\mathfrak{X}))$. Therefore we have $X = \mathcal{S}(\mathfrak{X}) \oplus \mathcal{DS}(\mathfrak{X})$. This completes the proof. $\square$

4. Conclusion

As we know, the primary aim of the theory of WFI-algebras is to determine the structure of all WFI-algebras. The main task of a structure theorem is to find a complete system of invariants describing the WFI-algebra up to isomorphism, or to establish some connection with other mathematics branches. In addition, the filter theory plays an important role in studying WFI-algebras, and some interesting results have been obtained by several authors. In this paper we investigate the theory of decompositions in WFI-algebras, which is a useful tool for exploring the structure of WFI-algebras. Now we consider the subdirect sum and the direct sum of a filter family of a WFI-algebra (see Theorems 3.11 and 3.13). In the future we will discuss the direct product and the subdirect product of a WFI-algebraic family.

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