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LOCAL REGULARITY OF THE STEADY STATE NAVIER-STOKES EQUATIONS NEAR BOUNDARY IN FIVE DIMENSIONS

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Abstract. We present a new regularity criterion for suitable weak solutions of the steady-state Navier-Stokes equations near boundary in dimension five. We show that suitable weak solutions are regular up to the boundary if the scaled $L^{\frac{5}{2}}-$norm of the solution is small near the boundary. Our result is also valid in the interior.

1. Introduction

In this paper, we study suitable weak solutions $(u,p):\Omega\to\mathbb{R}^5\times\mathbb{R}$ to the stationary Navier-Stokes equations in a bounded domain Ω in five dimensions

$$(1.1)\quad \begin{cases} -\Delta u+(u\cdot\nabla)u+\nabla p=f, & \operatorname{div} \mathrm{u}=0 \quad \text{in } \ \Omega,\\ \qquad\qquad u=0 & \qquad\qquad\quad \text{on } \ \partial\Omega.\end{cases}$$

Here f is an external force and Ω is a bounded domain with $\mathcal{C}^2$ boundary.

After existence of weak solutions was proved by Leray [11] and Hopf [8] for the time dependent case in dimension three (see also [10] for steady state case), regularity question has remained open. The five dimensional

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steady-state Navier-Stokes equations has some similarities, e.g. dimensionless quantities due to scaling invariance, to the three dimensional Navier-Stokes equations. Furthermore, it is of independent interest itself since five dimension is the smallest dimension where stationary Navier-Stokes equations is super-critical (compare to [6]). Partial regularity was proved in [14] and [7] for the interior and boundary cases, respectively. The main point in [14] and [7] is that if the scaling invariant quantity $\frac{1}{r}\int_{\Omega\cap B_{x,r}} |\nabla u|^2\,dx$, under the scaling $u(x) \to \lambda u(\lambda x)$, is sufficiently small for $x \in \bar{\Omega}$, then u is regular at x. This implies that the one-dimensional Hausdorff measure of possible singular set is zero (compare to [1] and [13] in three dimensional time dependent case). The five dimensional steady-state Navier-Stokes equations have been also studied in a number of papers (see e. g. [2, 3, 4, 5]).

For a point $x = (x', x_5) \in \mathbb{R}^5$ with $x' \in \mathbb{R}^4$ we denote

$$B_{x,r} = \{y \in \mathbb{R}^5 \,:\, |y - x| < r\}, \qquad D_{x',r} = \{y' \in \mathbb{R}^4 \,:\, \left|y' - x'\right| < r\}.$$

For $x \in \bar{\Omega}$, we use the notation $\Omega_{x,r} = \Omega \cap B_{x,r}$ for some $r > 0$. If $x = 0$, we drop x in the above notations, for instance $\Omega_{0,r}$ is abbreviated to Ω_r.

A solution u to (1.1) is said to be regular at $x \in \bar{\Omega}$ if $u \in L^\infty(\bar{\Omega}_{x,r})$ for some $r > 0$. In such case, x is called a regular point. Otherwise we say that u is singular at x and x is a singular point.

We first make some assumptions on the boundary of Ω.

Assumption 1.1. Suppose that Ω is a domain with $\mathcal{C}^2$ boundary such that the following is satisfied: For each point $x = (x', x_5) \in \partial\Omega$ there exist L and r_0 independent of x such that we can find a Cartesian coordinate system $\{y_i\}_{i=1}^5$ with the origin at x and a $\mathcal{C}^2$ function $\varphi : D_{r_0} \to \mathbb{R}$ satisfying

$$\Omega_{r_0} = \Omega \cap B_{x,r_0} = \left\{y = (y', y_5) \in B_{x,r_1} \,:\, y_5 > \varphi(y')\right\}$$

and

$$\varphi(0) = 0, \quad \nabla_y \varphi(0) = 0, \quad \sup_{D_{r_0}} \left|\nabla_y^2 \varphi\right| \le L.$$

Remark 1.2. The main condition on Assumption 1.1 is the uniform estimate of the $\mathcal{C}^2-$norms of the function φ for each $x \in \partial\Omega$. More precisely, there exists a sufficiently small r_1 with $r_1 < r_0$, where r_0 is the number in Assumption 1.1 such that for any $r < r_1$

$$\sup_{x\in\partial\Omega} \|\varphi\|_{\mathcal{C}^2(D_r)} \le L(1 + r + r^2). \tag{1.2}$$

This can be verified by the Taylor's formula. □

We recall a class of functions, Morrey type space, denoted by $M_{2,\gamma}(\Omega)$ for some $0 < \gamma \leq 2$ so that $f \in M_{2,\gamma}(\Omega)$ is equipped with the norm

$$\|f\|_{M_{2,\gamma}(\Omega)} = \left(\sup_{\Omega_{x,r}, x\in\bar{\Omega}} \frac{1}{r^{1+2\gamma}} \int_{\Omega_r} |f|^2 \, dx \right)^{\frac{1}{2}}.$$

We note that $M_{2,\gamma}(\Omega)$ contains $L^{\frac{5}{2-\gamma}}(\Omega)$.

The objective of this paper is to present a sufficient condition for the regularity of suitable weak solutions to (1.1) near the boundary. Suitable weak solutions will be defined in Definition 2.1 in next section. Our main result is that the smallness of $L^{\frac{5}{2}}-$norm of the velocity field near boundary implies regularity. Our main result reads as follows:

THEOREM 1.3. *Let u be a suitable weak solution of the steady-state Navier-Stokes equation in Ω with force $f \in M_{2,\gamma}$ for some $\gamma > 0$. Assume further that Ω is a bounded domain with $\mathcal{C}^2$ boundary satisfying Assumption 1.1. Suppose that $x \in \partial\Omega$. There exists $\epsilon > 0$ such that if*

$$\limsup_{r\to 0} \frac{1}{r^{\frac{5}{2}}} \int_{\Omega_{x,r}} |u|^{\frac{5}{2}} \, dy < \epsilon,$$

then u is regular at x.

REMARK 1.4. It is an open question whether or not the regularity criterion in Theorem 1.3 could be replaced by the following condition:

$$\limsup_{r\to 0} \frac{1}{r^{5-m}} \int_{\Omega_{x,r}} |u|^m \, dy < \epsilon, \qquad 1 \leq m < \frac{5}{2}.$$

We present the proof of Theorem 1.3 in next section. Our regularity criterion is also true in the interior, which will be shown in the Appendix.

2. Local boundary regularity

In this section we introduce the notations, define suitable weak solutions, derive the equations (2.3) changed by flatting the boundary, and finally present the proof of Theorem 1.3. We begin with some notations. Let Ω be a bounded domain in $\mathbb{R}^5$. We denote by $C = C(\alpha, \beta, ...)$ a constant depending on the prescribed quantities $\alpha, \beta, ...$, which may change from line to line. For $1 \leq q \leq \infty$, $W^{k,q}(\Omega)$ denotes the usual Sobolev space, i.e., $W^{k,q}(\Omega) = \{u \in L^q(\Omega) : D^\alpha u \in L^q(\Omega), 0 \leq |\alpha| \leq k\}$. We write the average of f on E as $\fint_E f$, that is $\fint_E f = \int_E f / |E|$. Next we recall suitable weak solutions for the steady-state Navier-Stokes equations (1.1) in five dimensions (compare to e.g. [1] for three dimensional case).

DEFINITION 2.1. Let $\Omega \subset \mathbb{R}^5$ be a bounded domain satisfying Assumption 1.1. Suppose that f belongs to the Morrey space $M_{2,\gamma}(\Omega)$ for some $\gamma > 0$. A pair (u,p) is suitable weak solution to (1.1) if the following conditions are satisfied:

(a) The functions $u : \Omega \to \mathbb{R}^5$ and $p : \Omega \to \mathbb{R}$ satisfy $u \in W_0^{1,2}(\Omega)$, $p \in L^{\frac{3}{2}}(\Omega)$.
(b) u and p solve the Navier-Stokes equations in Ω in the sense of distributions and u satisfies the boundary condition $u = 0$ on $\partial\Omega$ in the trace sense.
(c) u and p satisfy the local energy inequality

$$\int_\Omega |\nabla u|^2 \phi \le \frac{1}{2}\int_\Omega |u|^2 \Delta\phi + \int_\Omega \left(\frac{|u|^2}{2} + p\right) u \cdot \nabla\phi + \int_\Omega f \cdot u\phi, \tag{2.1}$$

where $\phi \in C_0^\infty(\mathbb{R}^5)$ and $\phi \ge 0$. □

REMARK 2.2. The existence of suitable weak solutions is proved in [1] (refer also to [12]) and Definition 2.1 for suitable weak solution for steady case is analog to that of time dependent case. The main difference between suitable weak solutions and weak solutions (compare to [2, p.779]) is the additional condition of the local energy inequality (2.1). It is an open question if all weak solutions are suitable.

Let $x_0 \in \partial\Omega$. Under Assumption 1.1, we can represent $\Omega_{x_0,r_0} = \Omega \cap B_{x_0,r_0} = \{y = (y', y_5) \in B_{x_0,r_0} : y_5 > \varphi(y')\}$, where φ is the graph of $\mathcal{C}^2$ in Assumption 1.1. Flatting the boundary near x_0, we introduce new coordinates $x = \psi(y)$ by formulas

$$x = \psi(y) \equiv (y_1,\, y_2,\, y_3,\, y_4,\, y_5 - \varphi(y_1, y_2, y_3, y_4))\,. \tag{2.2}$$

We note that the mapping $y \mapsto x = \psi(y)$ straightens out $\partial\Omega$ near x_0 such that $\Omega_{x_0,\rho}$, $\rho < r_0$ is transformed onto a subdomain $\psi(\Omega_{x_0,\rho})$ of $\mathbb{R}^5_+ \equiv \{x \in \mathbb{R}^5 : x_5 > 0\}$.

We define $v = u \circ \psi^{-1}, \pi = p \circ \psi^{-1}$ and $g = f \circ \psi^{-1}$ in $\psi(\Omega_{x_0,\rho})$. Then using the change of variables (2.2), the equations (1.1) result in the following for v and π:

$$\begin{cases} -\hat{\Delta} v + (v \cdot \hat{\nabla})v + \hat{\nabla}\pi = g & \text{in } \psi(\Omega_{x_0,\rho}), \\ \hat{\nabla} \cdot v = 0 & \text{in } \psi(\Omega_{x_0,\rho}), \\ v = 0 & \text{on } \partial\psi(\Omega_{x_0,\rho}) \cap \{x_5 = 0\}, \end{cases} \tag{2.3}$$

where $\hat{\nabla}$ and $\hat{\Delta}$ are differential operators with variable coefficients defined by

(2.4)
$$\hat{\nabla} = (\partial_{x_1} - \varphi_{x_1}\partial_{x_5}, \partial_{x_2} - \varphi_{x_2}\partial_{x_5}, \partial_{x_3} - \varphi_{x_3}\partial_{x_5}, \partial_{x_4} - \varphi_{x_4}\partial_{x_5}, \partial_{x_5}),$$
$$\hat{\Delta} = a_{ij}(x)\partial^2_{x_i,x_j} + b_i(x)\partial_{x_i},$$

where a_{ij} and b_i are given as

$$a_{ij}(x) = \delta_{ij}, \quad a_{i5}(x) = a_{5i}(x) = -\varphi_{x_i}, \quad b_i(x) = 0, \qquad i = 1, 2, 3, 4,$$

and

$$a_{55}(x) = 1 + \sum_{i=1}^{4}(\varphi_{x_i})^2, \qquad b_5(x) = -\sum_{i=1}^{4}\varphi_{x_i x_i}.$$

As mentioned in Remark 1.2, if we take a sufficiently small r_1 with $r_1 < r_0$, then (1.2) holds for any $r < r_1$. In addition, we note that

$$\text{(2.5)} \qquad \frac{1}{2}\left|\nabla v(x,t)\right| \le \left|\hat{\nabla} v(x,t)\right| \le 2\left|\nabla v(x,t)\right| \qquad \text{for all } x \in \psi(\Omega_{x_0,\rho})$$

and

(2.6)
$$B^+_{\psi(x_0),\frac{r}{2}} \subset \psi(\Omega_{x_0,r}) \subset B^+_{\psi(x_0),2r}, \quad \psi^{-1}(B^+_{\psi(x_0),\frac{r}{2}}) \subset \Omega_{x_0,r} \subset \psi^{-1}(B^+_{\psi(x_0),2r}),$$

where $B^+_{x,r}$ indicates $\{y \in \mathbb{R}^5_+ : |y - x| < r\}$.

From now on, we fix $x_0 = 0$ without loss of generality. We suppose that, as above, ψ is a coordinate transformation so that v, π satisfies (2.3) in $\psi(\Omega_{r_0})$.

Remark 2.3. Due to the suitability of u, p (see Definition 2.1), (v, π) solve (2.3) in a weak sense and satisfies the following local energy inequality: There exists r_2 with $r_2 < r_0$ where r_0 is the number in Assumption 1.1 such that

(2.7)
$$\int_{\psi(\Omega_{r_0})} \left|\hat{\nabla} v\right|^2 \xi \le C\int_{\psi(\Omega_{r_0})} \left(|v|^2\left|\hat{\Delta}\xi\right| + (|v|^3 + |\pi|^{\frac{3}{2}})\left|\hat{\nabla}\xi\right| + |g\cdot v|\,|\xi|\right),$$

where $\xi \in C_0^\infty(B_r)$ with $r < r_2$ and $\xi \ge 0$, and $\hat{\nabla}$ and $\hat{\Delta}$ are differential operators in (2.4). □

Next we define some scaling invariant functionals, which are useful for our purpose. Let $B_r^+ = B_r \cap \{x \in \mathbb{R}^5 : x_5 > 0\}$. Let r_0 and r_1 be the numbers in Assumption 1.1 and Remark 1.2, respectively. For any $r < r_1$ and for a suitable weak solution (u, p) of (1.1) we introduce

$$E(r) := \frac{1}{r}\int_{\Omega_r} |\nabla u(y)|^2 dy, \quad A(r) := \frac{1}{r^2}\int_{\Omega_r} |u(y)|^3 dy,$$

$$K(r) := \frac{1}{r^{\frac{5}{2}}}\int_{\Omega_r} |u(y)|^{\frac{5}{2}} dy, \quad Q(r) := \frac{1}{r^2}\int_{\Omega_r} |p(y)|^{\frac{3}{2}} dy.$$

For a weak solution (v, π) and $B_r^+ \subset \psi(\Omega_{r_1})$, we introduce

$$\hat{E}(r) := \frac{1}{r}\int_{B_r^+} |\hat{\nabla} v(y)|^2 dy, \quad \hat{A}(r) := \frac{1}{r^2}\int_{B_r^+} |v(y)|^3 dy,$$

$$\hat{K}(r) := \frac{1}{r^{\frac{5}{2}}}\int_{B_r^+} |v(y)|^{\frac{5}{2}} dy, \quad \hat{Q}(r) := \frac{1}{r^2}\int_{B_r^+} |\pi(y)|^{\frac{3}{2}} dy.$$

Next lemma shows relations between scaling invariant quantities above.

Lemma 2.4. *Let Ω be a bounded domain satisfying Assumption 1.1 and $x_0 \in \partial\Omega$. Suppose that (u, p) and (v, π) are suitable weak solutions of* (1.1) *in $\Omega \subset \mathbb{R}^5$ and* (2.3) *in $\psi(\Omega_{x_0}) \subset \mathbb{R}^5_+$, respectively, where ψ is the mapping flatting the boundary in the Assumption 1.1. Then there exist sufficiently small r_1 and $C = C(r_1)$ such that for any $4r < r_1$ the followings are satisfied:*

$$\frac{1}{C}E(r) \le \hat{E}(2r) \le CE(4r), \qquad \frac{1}{C}A(r) \le \hat{A}(2r) \le CA(4r),$$
$$\frac{1}{C}K(r) \le \hat{K}(2r) \le CK(4r), \qquad \frac{1}{C}Q(r) \le \hat{Q}(2r) \le CQ(4r),$$

Proof. We just show one of above estimates, since others follows similar arguments. As indicated earlier, we take a sufficiently small r_1 such that (1.2), (2.5) and (2.6) hold. Then

$$E(r) \le \frac{C}{r}\int_{\psi(\Omega_r)} |\nabla v|^2 \le \frac{C}{r}\int_{\psi(\Omega_r)} \left|\hat{\nabla} v\right|^2 \le \frac{C}{2r}\int_{B_{2r}^+} \left|\hat{\nabla} v\right|^2 = C\hat{E}(2r).$$

On the other hand,

$$\hat{E}(2r) \le \frac{1}{2r}\int_{B_{2r}^+} |\nabla v|^2 \le \frac{C}{r}\int_{\psi^{-1}(B_{2r}^+)} |\nabla u|^2 \le \frac{C}{4r}\int_{\Omega_{4r}} |\nabla u|^2 = CE(4r).$$

This completes the proof. □

Remark 2.5. We can also show that f and g have relations as in Lemma 2.4. To be more precise, for $1 \le m < \infty$

$$\int_{\Omega_r} |f|^m \le C\int_{\psi(\Omega_r)} |g|^m \le C\int_{B_{2r}^+} |g|^m,$$

$$\int_{B_r^+} |g|^m \le C \int_{\psi^{-1}(B_r^+)} |f|^m \le C \int_{\Omega_{2r}} |f|^m .$$

Therefore, it is direct that $\|g\|_{M_{2,\gamma}(B_r^+)} \le C \|f\|_{M_{2,\gamma}(\Omega)}$. □

From now on, for simplicity, we denote $\|f\|_{M_{2,\gamma}} = m_\gamma$. Next we show the local regularity criterion for v near the boundary.

LEMMA 2.6. *Let Ω be a bounded domain satisfying the Assumption 1.1 and $x_0 \in \partial\Omega$. Suppose that (u,p) and (v,π) are suitable weak solutions of (1.1) in $\Omega \subset \mathbb{R}^5$ and (2.3) in $\psi(\Omega_{x_0}) \subset \mathbb{R}^5_+$, respectively, where ψ is the mapping flatting the boundary in the Assumption 1.1. Let $x = \psi(x_0)$. Assume further that $g \in M_{2,\gamma}$ for some $\gamma \in (0,2]$. Then there exist $\epsilon > 0$ and r_* depending on $\gamma, \|g\|_{M_{2,\gamma}}$ such that if $\hat{A}^{\frac{1}{3}}(r) < \epsilon$ for some $r < r_*$, then x is a regular point.*

Before giving the proof of Lemma 2.6, we first control the pressure in terms of velocity field, which will be used in the proof of Lemma 2.6.

LEMMA 2.7. *Under the assumption in Lemma 2.6, there exists r_* such that for any r and ρ with $16r < \rho < r_*$*

$$\hat{Q}(r) \le C(\frac{\rho}{r})^2 \left(\hat{A}(\rho) + \rho^{\frac{3\gamma}{2}} m_\gamma^{\frac{3}{2}} \right).$$

Proof. Let r_* be sufficiently small such that (1.2), (2.5) and (2.6) hold. Due to the estimate of pressure for the Stokes system in [9, Theorem 3.7], we know

$$Q(2r) \le C \left(A(4r) + \frac{1}{r^2} \int_{\Omega_{4r}} |f|^{\frac{3}{2}} \right) \le \frac{C}{r^2} \left(\hat{A}(8r) + \int_{B_{8r}^+} |g|^{\frac{3}{2}} \right).$$

Combining the above estimate and Lemma 2.4, we obtain

$$\hat{Q}(r) \le CQ(2r) \le \frac{C}{r^2} \left(\hat{A}(8r) + \int_{B_{8r}^+} |g|^{\frac{3}{2}} \right)$$
$$\le C(\frac{\rho}{r})^2 \hat{A}(\rho) + C(\frac{\rho}{r})^2 \left(\rho^{-1} \int_{B_\rho^+} |g|^2 \right)^{\frac{3}{4}}.$$

Recalling observations in Remark 2.5, we deduce the lemma. This completes the proof. □

The proof of Lemma 2.6 is based on the following decay estimate of v in a Lebesgue spaces.

Lemma 2.8. *Let $0 < \theta < \frac{1}{2}$ and $\beta \in (0, \gamma)$. Under the assumption in Lemma 2.6, there exist $\epsilon_1 > 0$ and r_* depending on θ, γ, β and m_γ such that if $\hat{A}^{\frac{1}{3}}(r) + \|g\|_{M_\gamma} r^{\beta+1} < \epsilon_1$ for some $r \in (0, r_*)$, then*

$$\hat{A}^{\frac{1}{3}}(\theta r) < C\theta^{1+\alpha}\left(\hat{A}^{\frac{1}{3}}(r) + m_\gamma r^{\beta+1}\right),$$

where $0 < \alpha < 1$ and C is a constant.

Proof. Suppose the statement is not true. Then for any $\alpha \in (0,1)$, $\theta \in (0, \frac{1}{2})$, and $C > 0$, there exist $x_n \searrow 0, r_n \searrow 0$ and $\varepsilon_n \searrow 0$ such that $\hat{A}^{\frac{1}{3}}(r_n) + \|g_n\|_{M_\gamma} r_n^{\beta+1} = \varepsilon_n$, but $\hat{A}^{\frac{1}{3}}(\theta r_n) > C\theta^{1+\alpha}(\hat{A}^{\frac{1}{3}}(r_n) + m_\gamma r_n^{\beta+1})$. Let $y = r_n^{-1}(x - x_n)$ and we define w_n,q_n and h_n by $w_n(y) := \varepsilon_n^{-1} r_n v_n(x)$, $q_n(y) := \varepsilon_n^{-1} r_n^2 \pi_n(x)$ and $h_n(y) := \varepsilon_n^{-1} r_n^3 g_n(x)$ respectively. For convenience we denote $\hat{A}(w_n, \theta) := \frac{1}{\theta}\int_{B_1^+} |w_n|^3 dy$ and $M_\gamma^n := \|h_n\|_{M_{2,\gamma}}$. The change of variables leads to

$$\hat{A}^{\frac{1}{3}}(w_n, 1) + M_\gamma^n r_n^{\beta-\gamma} = 1 \quad \text{and} \quad \hat{A}^{\frac{1}{3}}(w_n, \theta) \geq C\theta^{1+\alpha}. \tag{2.8}$$

On the other hand, w_n, q_n solve the following system in a weak sense:

$$\begin{cases} -\hat{\Delta} w_n + \varepsilon_n (w_n \cdot \hat{\nabla}) w_n + \hat{\nabla} q_n = h_n, \quad \operatorname{div} w_n = 0 & \text{in } B_1^+ \\ w_n = 0 & \text{on } B_1 \cap \{x_5 = 0\}. \end{cases}$$

Due to (2.8) and Lemma 2.7, we have following weak convergence (possible subsequences of w_n and q_n should be taken and we, however, use the same symbol for simplicity):

$$w_n \rightharpoonup w \text{ in } L^3(B_1^+), \qquad q_n \rightharpoonup q \text{ in } L^{\frac{3}{2}}(B_{\frac{3}{4}}^+),$$

$$\|h_n\|_{L^{\frac{3}{2}}} \leq \varepsilon_n^{-1} M_\gamma^n r_n^{1+\gamma} = \varepsilon_n^{-1} M_\gamma^n r_n^{\beta+1} r_n^{\gamma-\beta} \leq r_n^{\gamma-\beta} \longrightarrow 0 \ \ as\ n \longrightarrow \infty.$$

Next we show that $\hat{\nabla} w_n$ is uniformly bounded in $L^2(B_{\frac{1}{2}}^+)$. Let ξ be a standard cut off function satisfying $\xi = 1$ on $B_{\frac{1}{2}}$ and $\xi = 0$ on $\mathbb{R}^5 \setminus B_{\frac{3}{4}}$. Recalling the local energy inequality,

$$\int_{B_1^+} |\hat{\nabla} w_n|^2 \xi dy \leq C \int_{B_1^+} |w_n|^2 \left|\hat{\Delta}\xi\right| dy + \int_{B_1^+} \left(|w_n|^3 + |q_n w_n|\right) \left|\hat{\nabla}\xi\right| + \int_{B_1^+} |h_n \cdot s_n||\xi| dy,$$

we have following weak convergence; $w_n \rightharpoonup w$ in $W^{1,2}(B_{\frac{3}{4}}^+)$. Thus, by the compactness theorem, we have strong convergence of w_n to w in

$L^3(B^+_{\frac{1}{2}})$, namely

$$w_n \to w \ \text{ in } \ L^3(B^+_{\frac{1}{2}}). \tag{2.9}$$

Combining (2.9) and $\hat{A}^{\frac{1}{3}}(w_n, \theta) > C\theta^{1+\alpha}$, we obtain

$$\left(\frac{1}{\theta^2} \int_{B^+_\theta} |w|^3 dy \right)^{\frac{1}{3}} \geq C\theta^{1+\alpha}. \tag{2.10}$$

We note that w and q solve the following Stokes system in a weak sense:

$$-\Delta w + \nabla q = 0, \qquad \text{div } w = 0 \ \text{ in } \ B^+_{\frac{3}{4}}, \qquad w = 0 \ \text{ on } \{x_5 = 0\} \cap B_{\frac{3}{4}}.$$

Since w is regular in $B^+_{\frac{1}{2}}$, we obtain

$$\hat{A}^{\frac{1}{3}}(w) = \left(\frac{1}{\theta^2} \int_{B^+_\theta} |w|^3 dy \right)^{\frac{1}{3}} \leq C_1\theta^2,$$

where C is absolute constant. At the beginning, we take $\theta \in (0, \frac{1}{2})$ such that $C_1\theta^2 \leq \frac{C}{2}\theta^{1+\alpha}$, where C is the constant in (2.10). This leads, due to (2.10) and the choice of θ, to a contradiction, because $C\theta^{1+\alpha} \leq \liminf \hat{A}^{\frac{1}{3}}(w_n, \theta) = \hat{A}^{\frac{1}{3}}(w, \theta) = C_1\theta^2 \leq \frac{C}{2}\theta^{1+\alpha}$. This completes the proof. □

Since the Lemma 2.8 is the crucial part of the proof of Lemma 2.6, we present only a brief sketch of the streamline of Lemma 2.6

The sketch of the proof of Lemma 2.6 We note that due to the Lemma 2.8 there exists a positive constant $\alpha_1 < 1$ such that

$$\hat{A}^{\frac{1}{3}}(r) < C\theta^{1+\alpha_1} \left(\hat{A}^{\frac{1}{3}}(\rho) + m_\gamma r^{\beta+1} \right), \qquad r < \rho < r_1,$$

where r_1 is the number in Lemma 2.6. We consider for any $x \in B^+_{r_1/2}$ and for any $r < r_1/4$

$$\hat{A}_a(r) = \frac{1}{r^2} \int_{B^+_x} |v(y) - (v)_a|^3 \, dy, \qquad (v)_a = \fint_{B^+_x} v(y) dy.$$

We can then show that $\hat{A}_a^{\frac{1}{3}}(r) \leq Cr^{1+\alpha_1}$, where C is an absolute constant independent of v. This can be proved by simple computations and the details are omitted. Hölder continuity of v is a direct consequence of this estimate, which immediately implies that u is also Hölder continuous locally near boundary. This completes the proof. □

In next lemma we estimate the scaled L^3-norm of suitable weak solutions.

Lemma 2.9. *Let r_* be the number in Lemma 2.6. Under the same assumption as in Lemma 2.6, for any $r < r_1$*

$$\hat{A}(r) \le C\hat{E}(r)\hat{K}^{\frac{2}{5}}(r). \tag{2.11}$$

Proof. This is due to the Hölder and Poincaré's inequality. Indeed, as before, taking a sufficiently small r_1 such that (2.5) holds, we have

$$\|v\|^3_{L^3(B_r^+)} \le C\|\nabla v\|^2_{L^2(B_r^+)}\|v\|_{L^{\frac{5}{2}}(B_r^+)} \le C\left\|\hat{\nabla} v\right\|^2_{L^2(B_r^+)}\|v\|_{L^{\frac{5}{2}}(B_r^+)}.$$

We deduce (2.11) by diving both sides by r^2. This completes the proof. □

Now we are ready to present the proof of Theorem 1.3.

The proof of Theorem 1.3 Let $\hat{r}$ be sufficiently small such that (1.2), (2.5) and (2.6) hold and we will specify $\hat{r}$ later. We assume that $16r < \rho < \hat{r}$. By Lemma 2.7 and Lemma 2.9, we have for $16r < \rho$

$$\begin{aligned}
\hat{A}(r) &\le C\hat{E}(r)\hat{K}^{\frac{2}{5}}(r)\\
&\le C\left[\hat{A}^{\frac{2}{3}}(2r) + \hat{A}(2r) + \hat{Q}(2r) + m_\gamma \hat{A}^{\frac{2}{3}}(2r) r^{\frac{3\gamma}{2}}\right]\hat{K}^{\frac{2}{5}}(r)\\
&\le C\left[\hat{A}(2r) + \hat{Q}(2r) + 1 + m_\gamma^3 r^{\frac{9\gamma}{2}}\right]\hat{K}^{\frac{2}{5}}(r)\\
&\le C(\frac{\rho}{r})^2\hat{K}^{\frac{2}{5}}(r)\hat{A}(\rho) + C(\frac{\rho}{r})^2 m_\gamma^{\frac{3}{2}}\rho^{\frac{3\gamma}{2}}\hat{K}^{\frac{2}{5}}(r) + C\left[1 + m_\gamma^3 r^{\frac{9\gamma}{2}}\right]\hat{K}^{\frac{2}{5}}(r).
\end{aligned}$$

Choosing $\theta \in (0, \frac{1}{4})$ and replacing r and ρ by θr and r, respectively, we obtain

$$\hat{A}(\theta r) \le \frac{C}{\theta^2}\hat{K}^{\frac{2}{5}}(\theta r)\hat{A}(r) + \frac{C}{\theta^2}m_\gamma^{\frac{3}{2}} r^{\frac{3\gamma}{2}}\hat{K}^{\frac{2}{5}}(\theta r) + C\left[1 + m_\gamma^3(\theta r)^{\frac{9\gamma}{2}}\right]\hat{K}^{\frac{2}{5}}(\theta r). \tag{2.12}$$

Now we fix $\hat{r} < \min\{1, (1+m_\gamma)^{-\frac{1}{\gamma}}, (1+m_\gamma)^{-\frac{2}{3\gamma}}, r_*\}$, where r_* is the number in Lemma 2.6 such that for all $r \le \hat{r}$

$$\hat{K}(r) < \min\left\{\left(\frac{\theta^2}{2C}\right)^{\frac{5}{2}}, \left(\frac{\theta^2\epsilon}{8C}\right)^{\frac{5}{2}}\right\},$$

where C is the constant in (2.12) and ϵ is the number in Lemma 2.6. Therefore,

$$\hat{A}(\theta r) \le \frac{1}{2}\hat{A}(r) + \frac{\epsilon}{4}.$$

By iteration, we obtain

$$\hat{A}(\theta^k r) \leq \frac{1}{2^k}\hat{A}(r) + \frac{\epsilon}{2}, \quad 4r < r_1.$$

Thus, for k sufficiently large, $\hat{A}(\theta^k r) < \epsilon$, which implies that x is a regular point due to Lemma 2.6. This completes the proof. □

Appendix

In this appendix we show that Theorem 1.3 is also valid in the interior case. Although interior case is simpler than the boundary case, we present a sketch of procedures since slight different arguments are required compared to the boundary case. From now on x is assumed to be an interior point and $B_{x,r} \subset \Omega$. We introduce a scaling invariant functional $A_a(r)$ defined by

$$A_a(r) := \frac{1}{r^2}\int_{B_{x,r}} |u(y) - (u)_r|^3\, dy, \qquad (u)_r := \fint_{B_{x,r}} u(y)dy.$$

In the case of interior, we need similar decay estimates comparable to Lemma 2.6 as in boundary case.

Lemma 2.10. *Let $0 < \theta < \frac{1}{2}, \beta \in (0,\gamma)$ and $x \in \Omega$ be an interior point. There exist $\varepsilon_1 > 0, r_1,$ and M depending on θ, γ, β such that if u is a suitable weak solution of Navier-Stokes equations satisfying Definition 2.1 and if $\hat{A}_a(r) + \|g\|_{M_r} r^{\beta+1} < \varepsilon_1$ and $|r(u)_r| < M$ for some $r \in (0, r_1)$, then*

$$A_a^{\frac{1}{3}}(\theta r) < C\theta^{1+\alpha}\left(A_a^{\frac{1}{3}}(r) + m_\gamma r^{\beta+1}\right),$$

where $0 < \alpha < 1$ and C is a constant.

Proof. We assume $f = 0$ for simplicity. Suppose the statement is not true. Then for any $\alpha \in (0,1)$ and $C > 0$, there exist $x_n, r_n \searrow 0$ and $\varepsilon_n \searrow 0$ such that $A_a^{\frac{1}{3}}(r_n) = \varepsilon_n$ and $r_n(u)_{r_n} < M$ but $A_a^{\frac{1}{3}}(\theta r_n) \geq C\theta^{1+\alpha}\epsilon_n$. Using the change of variables $y = r^{-1}(x - x_n)$, we set $v_n(y) := \varepsilon_n^{-1} r_n u(x)$ and $q_n = \varepsilon_n^{-1} r_n^2(p(x) - (p)_{r_n})$. The standard blow-up procedure and compactness result lead to a contradiction as in Lemma 2.6. Since the arguments are just repetitions of the boundary case, the details are skipped. □

Next lemma is straightforward due to Lemma 2.10, and thus its details are omitted.

LEMMA 2.11. *Let $x \in \Omega$ be an interior point. There exist a constant $\varepsilon > 0$ depending on γ, m_γ and $r_* > 0$ such that if u is a suitable weak solution of the Navier-Stokes equations satisfying Definition 2.1 and if $A^{\frac{1}{3}}(r) < \varepsilon$ for some $r < r_*$, then x is a regular point.*

With modifications above, we can have the same regularity criterion in the interior as in boundary case. Since its verification can be done by following procedures similar to those of boundary case, we skip the details.

THEOREM 2.12. *The same statement of Theorem 1.3 remains true when $x \in \Omega$ is an interior point with $\Omega_{x,r}$ replaced by $B_{x,r}$.*

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