JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **22**, No. 3, September 2009

SEVERAL RESULTS ASSOCIATED WITH THE RIEMANN ZETA FUNCTION

JUNESANG CHOI*

ABSTRACT. In 1859, Bernhard Riemann, in his epoch-making memoir, extended the Euler zeta function $\zeta(s)$ $(s > 1,\ s \in \mathbb{R})$ to the Riemann zeta function $\zeta(s)$ $(\Re(s) > 1,\ s \in \mathbb{C})$ to investigate the pattern of the primes. Sine the time of Euler and then Riemann, the Riemann zeta function $\zeta(s)$ has involved and appeared in a variety of mathematical research subjects as well as the function itself has been being broadly and deeply researched. Among those things, we choose to make a further investigation of the following subjects: Evaluation of $\zeta(2k)$ $(k \in \mathbb{N})$; Approximate functional equations for $\zeta(s)$; Series involving the Riemann zeta function.

1. Introduction and Preliminaries

We begin by recalling the definition of the Riemann zeta function $\zeta(s)$:

$$\zeta(s) := \begin{cases} \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1-2^{-s}} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^s} & (Re(s) > 1) \\ \displaystyle\left(1 - 2^{1-s}\right)^{-1} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} & (Re(s) > 0;\ s \neq 1). \end{cases} \tag{1.1}$$

This function $\zeta(s)$ has played an important role in the analytic number theory since Bernhard Riemann's epoch-making paper [15] entitled by *Über die Anzahl der Primzahlen unter einer gegebenen Grösse* (On the Number of Primes Less Than a Given Magnitude) whose an English translation is given in the Appendix of Edwards's book [10]. The

Received June 01, 2009; Accepted August 14, 2009.
2000 Mathematics Subject Classification: Primary 11M06, 11M99; Secondary 33B99.
Key words and phrases: Gamma function, Psi function, Polygamma function, Riemann zeta function, generalized zeta function, Bernoulli numbers and polynomials, Harmonic numbers, series involving the zeta function.

Hurwitz (or generalized) zeta function $\zeta(s,a)$ is defined by

$$\zeta(s,a) := \sum_{k=0}^{\infty} (k+a)^{-s} \qquad (Re(s) > 1,\ a \notin \mathbb{Z}_0^- := \{0,\ -1,\ -2,\ \ldots\}), \tag{1.2}$$

which, upon setting $a = 1$, yields the Riemann zeta function (1.1). It is noted that both of the Riemann zeta function $\zeta(s)$ and the Hurwitz zeta function $\zeta(s,a)$ can be continued meromorphically to the whole s-plane except for a simple pole only at $s = 1$ with their respective residue 1, in various ways.

The Riemann zeta function $\zeta(s)$ itself has been being broadly and deeply investigated. Moreover the function $\zeta(s)$ has involved and appeared in a variety of mathematical research subjects. For example,

$$\sum_{n=1}^{\infty} \frac{H_n}{(n+1)^2} = \tfrac{1}{2} \sum_{n=1}^{\infty} \frac{H_n}{n^2} = \zeta(3), \tag{1.3}$$

where $H_n := H_n^{(1)}$ denotes the harmonic numbers and $H_n^{(s)}$ denotes the generalized harmonic numbers defined by

$$H_n^{(s)} := \sum_{k=1}^{n} \frac{1}{k^s} \qquad (n \in \mathbb{N} := \{1,\ 2,\ 3,\ \ldots\};\ \ s \in \mathbb{C}). \tag{1.4}$$

Since the identity (1.3) was discovered by Euler in 1775 and has a long history (see, for example, [2, p. 252 *et seq.*]), a remarkably wide variety of summations whose terms are associated with the harmonic and generalized harmonic numbers has been evaluated, mainly, in terms of the Riemann zeta function $\zeta(s)$, under the research subject called *explicit evaluation of Euler sums* (see [9]). Among those diverse research subjects related to the Riemann zeta function $\zeta(s)$, we, here, are aiming mainly at presenting our small observations regarding the following subjects:

- Evaluation of $\zeta(2k)$ $(k \in \mathbb{N})$;
- Approximate functional equations for $\zeta(s)$;
- Series involving the Riemann zeta function.

For our purpose, we recall the following functions and polynomials: The *Bernoulli polynomials* $B_n(x)$ are defined by the generating function:

$$\frac{z\, e^{xz}}{e^z - 1} = \sum_{n=0}^{\infty} B_n(x)\, \frac{z^n}{n!} \quad (|z| < 2\pi). \tag{1.5}$$

The numbers $B_n := B_n(0)$ are called the *Bernoulli numbers* generated by

$$\frac{z}{e^z-1} = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!} \quad (|z| < 2\pi). \tag{1.6}$$

The Bernoulli polynomials and numbers $B_n(x)$ and B_n have many properties (see [19, Section 1.6]) three of which are recalled:

$$B_n(1-x) = (-1)^n B_n(x) \quad (n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}); \tag{1.7}$$

$$\zeta(-n,x) = -\frac{B_{n+1}(x)}{n+1} \quad (n \in \mathbb{N}_0); \tag{1.8}$$

$$\zeta(-n) = \begin{cases} -\dfrac{1}{2} & (n=0) \\ -\dfrac{B_{n+1}}{n+1} & (n \in \mathbb{N}). \end{cases} \tag{1.9}$$

The polygamma functions $\psi^{(n)}(z)$ of order n ($n \in \mathbb{N}_0$) are defined by (see [19, p. 22, Eq. (52)])

$$\psi^{(n)}(z) := \frac{d^{n+1}}{dz^{n+1}} \log \Gamma(z) = \frac{d^n}{dz^n}\psi(z) \quad (n \in \mathbb{N}_0, \quad z \notin \mathbb{Z}_0^-), \tag{1.10}$$

where $\psi(z) := \psi^{(0)}(z) = \Gamma'(z)/\Gamma(z)$ is called the psi (or digamma) function, and $\Gamma(z)$ is the well-known Gamma function.

2. Evaluation of $\zeta(2k)$ $(k \in \mathbb{N})$

The solution of the so-called *Basler problem* (*cf.*, *e.g.*, Spiess [17, p. 66]):

$$\zeta(2) = \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6} \tag{2.1}$$

was first found in 1736 by Leonhard Euler (1707-1783), although Jakob Bernoulli (1654-1705) and Johann Bernoulli (1667-1748) did their utmost to sum the series in (2.1). In fact, the former of these Bernoulli brothers did not live to see the solution of the problem, and the solution became known to the latter soon after Euler found (see, for details, Knopp [13, p. 238]).

Numerous interesting solutions of the problem of evaluating the Riemann $\zeta(2k)$ $(k \in \mathbb{N})$ have appeared in the mathematical literature ever since Euler first evaluated $\zeta(2)$. Here we recall two main formulas for evaluation of $\zeta(2k)$ $(k \in \mathbb{N})$ (see [19, p. 98]):

$$\zeta(2k) = (-1)^{k+1} \frac{(2\pi)^{2k}}{2\,(2k)!} B_{2k} \quad (k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}), \tag{2.2}$$

where B_k are Bernoulli numbers (see [19, Section 1.6]) which enables us to list the following special values:

$$\zeta(4) = \frac{\pi^4}{90}, \quad \zeta(6) = \frac{\pi^6}{945}, \quad \zeta(8) = \frac{\pi^8}{9450}, \quad \zeta(10) = \frac{\pi^{10}}{93555}, \ldots; \tag{2.3}$$

$$\zeta(2k) = \frac{2}{2k+1} \sum_{j=1}^{k-1} \zeta(2j)\zeta(2k-2j) \quad (k \in \mathbb{N} \setminus \{1\}), \tag{2.4}$$

which can also be used to evaluate $\zeta(2k)$ $(k \in \mathbb{N} \setminus \{1\})$ by starting with (2.1).

Don Zagier [21] described a short outline of an elementary proof of an equivalent form of (2.4):

$$\sum_{\substack{0<j<k \\ j \text{ even}}} \zeta(j)\,\zeta(k-j) = \frac{k+1}{2}\,\zeta(k) \quad (k \in \mathbb{N},\ k \geq 4 \text{ even}). \tag{2.5}$$

Here we give a rather detailed proof of (2.5) by complementing that of Zagier's. Indeed, we prove

$$f_k(m,n) - f_k(m+n,n) - f_k(m,m+n) = \sum_{\substack{0<j<k \\ j \text{ even}}} \frac{1}{m^j\, n^{k-j}}, \tag{2.6}$$

where $f_k(m,n)$ is defined by

$$f_k(m,n) := \frac{1}{m \cdot n^{k-1}} + \frac{1}{2} \sum_{r=2}^{k-2} \frac{1}{m^r\, n^{k-r}} + \frac{1}{m^{k-1}\, n}.$$

The proof is proceeded by induction on k. Starting with $k = 4$, it can be directly checked

$$f_4(m,n) - f_4(m+n,n) - f_4(m,m+n) = \frac{1}{m^2\, n^2},$$

which, upon summing over all integers m, $n > 0$, yields

$$\zeta(2)^2 = \left(\sum_{m,\, n>0} - \sum_{m>n>0} - \sum_{n>m>0} \right) f_4(m,n)$$
$$= \sum_{n>0} f_4(n,n) = \frac{5}{2}\,\zeta(4).$$

Thus the formula (2.1) gives $\zeta(4)$ in (2.3). For convenience, let

$$I_{m,n}(k) := f_k(m,n) - f_k(m+n,n) - f_k(m,m+n).$$

Then it is found that

$$I_{m,n}(k) = \sum_{\substack{0<j<k \\ j \text{ even}}} \frac{1}{m^j\, n^{k-j}} + \mathcal{A}_{m,n}(k),$$

where

$$\mathcal{A}_{m,n}(k) : = \frac{1}{2} \sum_{\substack{j=2 \\ j \text{ even}}}^{k-2} \frac{1}{m^j\, n^{k-j}} - \frac{1}{2} \sum_{\substack{j=2 \\ j \text{ odd}}}^{k-2} \frac{1}{m^j\, n^{k-j}}$$
$$- \frac{1}{m \cdot n \cdot (m+n)^{k-2}} - \frac{1}{2} \sum_{r=2}^{k-2} \frac{m^r + n^r}{m^r\, n^r\, (m+n)^{k-r}}.$$

Now it is sufficient to show that $\mathcal{A}_{m,n}(k) = 0$ for all even $k \in \mathbb{N}$, $k \geq 4$, or, equivalently,

$$\mathcal{P}_{m,n}(k) = \mathcal{Q}_{m,n}(k) \quad (k \in \mathbb{N},\ k \geq 4 \text{ even}), \tag{2.7}$$

where

$$\mathcal{P}_{m,n}(k) := \sum_{\substack{j=2 \\ j \text{ even}}}^{k-2} \frac{1}{m^j\, n^{k-j}} - \sum_{\substack{j=2 \\ j \text{ odd}}}^{k-2} \frac{1}{m^j\, n^{k-j}}$$

and

$$\mathcal{Q}_{m,n}(k) := \frac{2}{m \cdot n \cdot (m+n)^{k-2}} + \sum_{r=2}^{k-2} \frac{m^r + n^r}{m^r\, n^r\, (m+n)^{k-r}}.$$

When $k = 4$ is already checked. Assume that (2.7) holds true for some even $k \in \mathbb{N}$, $k \geq 4$. We have

$$\mathcal{P}_{m,n}(k+2) = \sum_{\substack{j=2 \\ j \text{ even}}}^{k} \frac{1}{m^j\, n^{k+2-j}} - \sum_{\substack{j=2 \\ j \text{ odd}}}^{k} \frac{1}{m^j\, n^{k+2-j}},$$

which, upon $j-2=j'$ and then dropping the prime on j, becomes

$$\begin{aligned}
\mathcal{P}_{m,n}(k+2) &= \sum_{\substack{j=0 \\ j \text{ even}}}^{k-2} \frac{1}{m^{j+2}\, n^{k-j}} - \sum_{\substack{j=0 \\ j \text{ odd}}}^{k-2} \frac{1}{m^{j+2}\, n^{k-j}} \\
&= \frac{1}{m^2} \left(\sum_{\substack{j=0 \\ j \text{ even}}}^{k-2} \frac{1}{m^{j}\, n^{k-j}} - \sum_{\substack{j=0 \\ j \text{ odd}}}^{k-2} \frac{1}{m^{j}\, n^{k-j}} \right) \\
&= \frac{1}{m^2} \left\{ \left(\frac{1}{n^k} - \frac{1}{m \cdot n^{k-1}} \right) + \sum_{\substack{j=2 \\ j \text{ even}}}^{k-2} \frac{1}{m^{j}\, n^{k-j}} - \sum_{\substack{j=2 \\ j \text{ odd}}}^{k-2} \frac{1}{m^{j}\, n^{k-j}} \right\}.
\end{aligned}$$

By induction hypothesis, we, therefore, get

$$\begin{aligned}
&\mathcal{P}_{m,n}(k+2) \\
&= \frac{1}{m^2} \left\{ \left(\frac{1}{n^k} - \frac{1}{m \cdot n^{k-1}} \right) + \frac{2}{m \cdot n \cdot (m+n)^{k-2}} + \sum_{r=2}^{k-2} \frac{m^r + n^r}{m^r n^r (m+n)^{k-r}} \right\},
\end{aligned}$$

which, upon summing the finite geometric series, gives

$$\begin{aligned}
&\mathcal{P}_{m,n}(k+2) \\
&= \frac{1}{m^2} \left\{ \left(\frac{1}{n^k} - \frac{1}{m \cdot n^{k-1}} \right) + \frac{1}{m \cdot n^{k-2} \cdot (m+n)} + \frac{1}{n \cdot m^{k-2} \cdot (m+n)} \right\} \\
&= \frac{1}{m \cdot n^k \cdot (m+n)} + \frac{1}{n \cdot m^k \cdot (m+n)}.
\end{aligned}$$

Similarly it can be shown that

$$\begin{aligned}
\mathcal{Q}_{m,n}(k+2) &= \frac{2}{m \cdot n \cdot (m+n)^k} + \sum_{r=2}^{k} \frac{m^r + n^r}{m^r\, n^r\, (m+n)^{k+2-r}} \\
&= \frac{1}{m \cdot n^k \cdot (m+n)} + \frac{1}{n \cdot m^k \cdot (m+n)}.
\end{aligned}$$

We thus have $\mathcal{P}_{m,n}(k+2) = \mathcal{Q}_{m,n}(k+2)$. So, by the principle of mathematical induction, (2.7) (and (2.6)) holds for all even $k \in \mathbb{N}$, $k \geq 4$. Summing over all integers m, $n > 0$ in (2.6), we finally proves the desired formula (2.5).

3. Approximate Functional Equations for $\zeta(s)$

There are a lot of practical situations which are necessary to deal with approximate functional equations for $\zeta(s)$. Many authors have been concerned to establish certain approximation formulas for $\zeta(s)$ to use them according to their respective necessities. For example, we recall an interesting formula with the error term given by a contour integral [12, p. 99]:

$$\zeta(s) = \sum_{n=1}^{m} n^{-s} + \frac{e^{-\pi i s}\,\Gamma(1-s)}{2\pi i} \int_C \frac{z^{s-1}\, e^{-m\,z}}{e^z - 1}\, dz, \tag{3.1}$$

where the contour C is essentially a Hankel's loop (*cf.*, *e.g.*, Whittaker and Watson [20, p. 245]), which starts from ∞ along the upper side of the positive real axis, encircles the origin once in the positive (counter-clockwise) direction, excluding the points $\pm 2k\pi i$ ($k \in \mathbb{N}$), and then returns to ∞ along the lower side of the positive real axis, as in Figure 1.

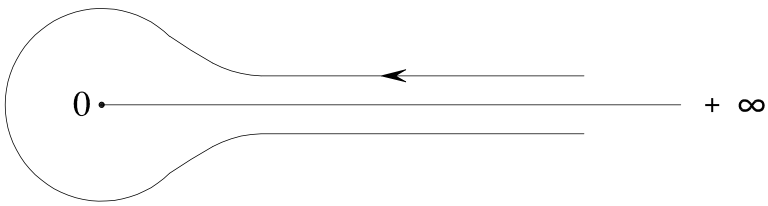

FIGURE 1

Among other approximate functional equations for $\zeta(s)$, we refer to another deeper formula due to Riemann and Siegel (see [12, Eq. (4.3), pp. 98–99]).

In view of the principle of deformation of paths (see [4, p. 159]), the loop C can be composed of three parts C_1, C_2, and C_3, where C_2 is a positively-oriented circle of radius δ ($0 < \delta < 2\pi$) about the origin, and C_1 and C_3 are the upper and lower edges of a cut in the complex z-plane along the positive real axis, traversed as described above, as in Figure 2:

It is interesting to observe that a well-known formula can be obtained from (3.1). If s is any integer in (3.1), the integrand in the contour integral in (3.1) takes the same values on both C_1 and C_3 with opposite signs, and hence the integrals along C_1 and C_3 cancel. So, setting $s = -k$

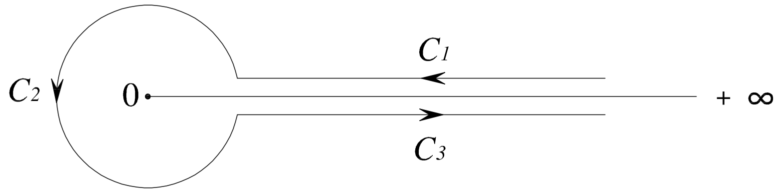

FIGURE 2

($k \in \mathbb{N}$) in (3.1) gives

$$\begin{aligned}\zeta(-k) &= \sum_{n=1}^{m} n^k + \frac{e^{\pi i k}\,\Gamma(1+k)}{2\pi i} \int_{C_2} \frac{z^{-k-1}\, e^{-m\,z}}{e^z - 1}\, dz \\ &= \sum_{n=1}^{m} n^k + (-1)^k\, k!\, \operatorname*{Res}_{z=0} f(z),\end{aligned}$$

where, for convenience,

$$f(z) := \frac{z^{-k-1}\, e^{-m\,z}}{e^z - 1} = z^{-k-2}\, \frac{z\, e^{-m\,z}}{e^z - 1}.$$

By virtue of (1.5), it is seen that

$$\operatorname*{Res}_{z=0} f(z) = \frac{B_{k+1}(-m)}{(k+1)!}.$$

we, thus, obtain an interesting formula

$$\sum_{n=1}^{m} n^k = \zeta(-k) - \frac{(-1)^k}{k+1}\, B_{k+1}(-m) \quad (m \in \mathbb{N},\ k \in \mathbb{N}). \tag{3.2}$$

If (1.7) and (1.9) is used in (3.2), a well-known desired formula equivalent to (3.2) is seen to be yielded (see [19, Eq.(17), p. 60]):

$$\sum_{n=1}^{m} n^k = \frac{B_{k+1}(m+1) - B_{k+1}}{k+1} \quad (m \in \mathbb{N},\ k \in \mathbb{N}). \tag{3.3}$$

The Bernoulli numbers are named after Jakob Bernoulli, who mentioned the numbers in his posthumous *Ars conjectandi* (see [3]). He discussed sums of equal powers of the first m integers in (3.3). The Bernoulli numbers appears in practically every field of mathematics, particularly, in combinatorial theory, finite difference calculus, numerical analysis, analytic number theory, and probability theory.

By using the Euler-Maclaurin summation formula: (*cf.* Hardy [11, p. 318]):

$$\sum_{k=1}^{n} f(k) \sim C_0 + \int_a^n f(x)\,dx + \frac{1}{2}\,f(n) + \sum_{r=1}^{\infty} \frac{B_{2r}}{(2r)!} f^{(2r-1)}(n), \tag{3.4}$$

where C_0 is an arbitrary constant to be determined in each special case and B_r are the Bernoulli numbers in (1.6), we can obtain a number of analytical representations of $\zeta(s)$, such as (*cf.* Hardy [11, p. 333])

$$\zeta(s) = \lim_{n\to\infty} \left\{ \sum_{k=1}^{n} k^{-s} - \frac{n^{1-s}}{1-s} - \frac{1}{2} n^{-s} \right\} \quad (\Re(s) > -1), \tag{3.5}$$

$$\zeta(s) = \lim_{n\to\infty} \left\{ \sum_{k=1}^{n} k^{-s} - \frac{n^{1-s}}{1-s} - \frac{1}{2} n^{-s} + \frac{1}{12} s n^{-s-1} \right\} \quad (\Re(s) > -3), \tag{3.6}$$

and

$$\begin{aligned} \zeta(s) = \lim_{n\to\infty} \Bigg\{ & \sum_{k=1}^{n} k^{-s} - \frac{n^{1-s}}{1-s} - \frac{1}{2} n^{-s} + \frac{1}{12} s n^{-s-1} \\ & - \frac{1}{720} s(s+1)(s+2) n^{-s-3} \Bigg\} \quad (\Re(s) > -5). \end{aligned} \tag{3.7}$$

Choi and Srivastava [8] (see also [19, p. 99–100]) used (3.6) and (3.7) to express mathematical constants B and C (which arise naturally in the study of multiple Gamma functions) defined by

$$\begin{aligned} \log B &= \lim_{n\to\infty} \left\{ \sum_{k=1}^{n} k^2 \log k - \left(\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6} \right) \log n + \frac{n^3}{9} - \frac{n}{12} \right\} \\ &\cong 0.03052113\ldots \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} \log C &= \lim_{n\to\infty} \left\{ \sum_{k=1}^{n} k^3 \log k - \left(\frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4} - \frac{1}{120} \right) \log n + \frac{n^4}{16} - \frac{n^2}{12} \right\} \\ &\cong -0.02065438\ldots \end{aligned} \tag{3.9}$$

in terms of the Riemann zeta function $\zeta(s)$ as follows:

$$\log B = -\zeta'(-2) \quad \text{and} \quad \log C = -\zeta'(-2) - \frac{11}{720}. \tag{3.10}$$

An application of Euler-Maclaurin summation formula (3.4) is made for the function

$$f(x) = x^{-s}\ (x > 0) \quad \text{and} \quad a = 1$$

to yield

$$\tag{3.11} \begin{aligned} \sum_{k=1}^{n} k^{-s} \sim C(s) + \frac{n^{1-s}}{1-s} + \frac{1}{s-1} + \frac{1}{2}\, n^{-s} \\ - \sum_{r=1}^{m} \frac{B_{2r}}{(2r)!}\, (s)_{2r-1}\, n^{-s-2r+1} + \mathcal{R}(s;\, m,\, n) \quad (n \to \infty), \end{aligned}$$

where $(s)_r := \Gamma(s+r)/\Gamma(s)$ $(r \in \mathbb{N}_0)$ is the Pochhammer symbol and $C(s)$ is a constant to be determined.

In order to determine the constant $C(s)$ in (3.11), it is assumed that $\Re(s) > 1$ and

$$\tag{3.12} C(s) = \lim_{n \to \infty} \left\{ \sum_{k=1}^{n} k^{-s} - \frac{n^{1-s} - 1}{1-s} \right\} = \zeta(s) + \frac{1}{1-s}.$$

From (3.11) and (3.12) we obtain a general asymptotic formula for $\zeta(s)$:

$$\tag{3.13} \begin{gathered} \zeta(s) = \lim_{n \to \infty} \left\{ \sum_{k=1}^{n} k^{-s} - \frac{n^{1-s}}{1-s} - \frac{1}{2\, n^{s}} + \sum_{r=1}^{m} \frac{B_{2r}}{(2r)!}\, (s)_{2r-1}\, n^{-s-2r+1} \right\} \\ (m \in \mathbb{N},\ \Re(s) > -2m-1,\ s \neq 1;\ m = 0,\ \Re(s) > 1) \end{gathered}$$

It is noted that formulas (3.5)–(3.7) are obvious special cases of the formula (3.13).

4. Series involving the Riemann Zeta function $\zeta(s)$

A classical (about three centuries old) theorem of Christian Goldbach (1690–1764), which was stated in a letter dated 1729 from Goldbach to Daniel Bernoulli (1700–1782), was revived in 1986 by Shallit and Zikan [16] as the following problem:

$$\tag{4.1} \sum_{\omega \in \mathcal{S}} (\omega - 1)^{-1} = 1,$$

where $\mathcal{S}$ denotes the set of all nontrivial integer kth powers, that is,

$$\tag{4.2} \mathcal{S} := \left\{ n^k \mid n, k \in \mathbb{N} \setminus \{1\} \right\}.$$

In terms of the Riemann Zeta function $\zeta(s)$ defined by (1.1), Goldbach's theorem (4.1) assumes the elegant form (*cf.* Shallit and Zikan [16, p. 403]):

$$\sum_{k=2}^{\infty} \{\zeta(k) - 1\} = 1 \tag{4.3}$$

or, equivalently,

$$\sum_{k=2}^{\infty} \mathcal{F}(\zeta(k)) = 1, \tag{4.4}$$

where $\mathcal{F}(x) := x - [x]$ denotes the *fractional* part of $x \in \mathbb{R}$. As a matter of fact, it is fairly straightforward to observe also that

$$\sum_{k=2}^{\infty} (-1)^k \mathcal{F}(\zeta(k)) = \frac{1}{2}, \tag{4.5}$$

$$\sum_{k=1}^{\infty} \mathcal{F}(\zeta(2k)) = \frac{3}{4}, \quad \text{and} \quad \sum_{k=1}^{\infty} \mathcal{F}(\zeta(2k+1)) = \frac{1}{4}. \tag{4.6}$$

The research subject of evaluating series such as (4.3)–(4.6) is referred to as closed-form evaluation of series involving the Riemann zeta function $\zeta(s)$ (or several other generalized zeta functions). This subject has been studied by many authors who have used a variety of methods and techniques (see, *e.g.*, [19, Chapter 3], [18], [7], [8]). Here we present more closed-form evaluation formulas by making use of known formulas.

Choi and Cvijović (see [5, Theorem 1]) proved a formula for $\psi^{(n)}(z)$ at rational arguments z: In terms of the Bernoulli polynomials $B_n(x)$ (see [19, Section 1.6]) and the generalized zeta functions $\zeta(s,a)$, they have:

$$\begin{aligned} \psi^{(n)}\left(\frac{p}{q}\right) &= (-1)^{n+1}\, n!\, q^n \\ &\cdot \sum_{s=0}^{q-1} \left\{ \mathcal{E}_n(s;p\,;q)\,(-1)^{1+\left\lfloor \frac{1}{2}(n+1)\right\rfloor} \frac{(2\pi)^{n+1}}{2\cdot(n+1)!}\, B_{n+1}\left(\frac{s}{q}\right) \right. \\ &\left. + \frac{1}{q^{n+1}}\, \mathcal{F}_n(s;p\,;q) \sum_{k=1}^{q} \zeta\left(n+1, \frac{k}{q}\right) \mathcal{E}_{n+1}(k;s\,;q) \right\} \end{aligned} \tag{4.7}$$

$$(p,\, q,\, n \in \mathbb{N};\ 1 \leq p < q),$$

where $\lfloor x \rfloor$ denotes the greatest integer $\leq x$, and

$$\mathcal{E}_n(s;p\,;q) := \frac{1+(-1)^n}{2}\,\sin\left(\frac{2\pi sp}{q}\right) + \frac{1-(-1)^n}{2}\,\cos\left(\frac{2\pi sp}{q}\right)$$

and

$$\mathcal{F}_n(s;p\,;q) := \frac{1+(-1)^n}{2}\,\cos\left(\frac{2\pi sp}{q}\right) + \frac{1-(-1)^n}{2}\,\sin\left(\frac{2\pi sp}{q}\right).$$

By using a known formula for $\psi^{(n)}(z)$ (see [19, p. 22, Eq. (55)]) and the series representation for $\cot z$, we get the following formula

$$\begin{aligned}\psi^{(n)}(z) - (-1)^n\,\psi^{(n)}(1-z) &= -\pi\frac{d^n}{dz^n}\{\cot\,\pi z\} \\ &= \frac{(-1)^{n+1}\,n!}{z^{n+1}} + 2\sum_{k=\lfloor\frac{n}{2}\rfloor+1}^{\infty}(2k-n)_n\,\zeta(2k)\,z^{2k-n-1} \end{aligned} \tag{4.8}$$

$$(n\in\mathbb{N}_0;\;\; 0<|z|<1)\,.$$

It is interesting to compare (4.8) with Eq. (25) in [19, p. 155] (see also Adamchik and Srivastava [1]).

Now we obtain the following results from (4.7) and (4.8): A closed-form evaluation of the following classes of series involving the Riemann zeta function $\zeta(s)$ is given:

(4.9)

$$\begin{aligned}&\sum_{k=n+1}^{\infty}(2k-2n)_{2n}\,\zeta(2k)\,\left(\frac{p}{q}\right)^{2k-2n-1} \\ &= \frac{(2n)!}{2}\,\left(\frac{q}{p}\right)^{2n+1} + (-1)^{n+1}\,\frac{(2\pi)^{2n+1}}{4}\sum_{s=0}^{q-1}s^{2n}\,\sin\left(\frac{2\pi sp}{q}\right) \\ &\quad + (-1)^n\,\frac{(2\pi)^{2n+1}\,q^{2n}}{2(2n+1)}\sum_{s=0}^{q-1}\sin\left(\frac{2\pi sp}{q}\right)\sum_{j=0}^{n}\binom{2n+1}{2j}B_{2j}\,\left(\frac{s}{q}\right)^{2n+1-2j}\end{aligned}$$

$$(p,\,q,\,n\in\mathbb{N};\;\;1\leq p<q)$$

and

(4.10)

$$\sum_{k=n}^{\infty} (2k+1-2n)_{2n-1}\,\zeta(2k)\,\left(\frac{p}{q}\right)^{2k-2n}$$

$$= -\frac{(2n-1)!}{2}\,\left(\frac{q}{p}\right)^{2n} + (-1)^n\,\frac{(2\pi)^{2n}}{4}\sum_{s=0}^{q-1} s^{2n-1}\,\cos\left(\frac{2\pi sp}{q}\right)$$

$$+ (-1)^{n+1}\,\frac{(2\pi)^{2n}\,q^{2n-1}}{4n}\sum_{s=0}^{q-1}\cos\left(\frac{2\pi sp}{q}\right)\sum_{j=0}^{n}\binom{2n}{2j}\,B_{2j}\,\left(\frac{s}{q}\right)^{2n-2j}$$

$$(p,\,q,\,n \in \mathbb{N};\ 1 \leq p < q).$$

References

[1] V. S. Adamchik and H. M. Srivastava, *Some series of the Zeta and related functions*, Analysis **18** (1998), 131–144.

[2] B. C. Berndt, *Ramanujan's Notebooks*, Part I, Springer-Verlag, New York, Berlin, Heidelberg, and Tokyo, 1985.

[3] J. Bernoulli, *Ars Conjectandi*, Basel, 1713. Reprinted on pp. 106–286 in Vol. 3 of *Die Werke von Jakob Bernoulli*, Birkhaüser Verlag, Basel, 1975, Academic Press, New York.

[4] J. W. Brown and R. V. Churchill, *Complex Variables and Applications*, 8th Edi., McGraw-Hill, Inc., New York, 2009.

[5] J. Choi and D. Cvijović, *Values of the polygamma functions at rational arguments*, J. Phys. A: *Math. Theor.* **40** (2007), 15019–15028.

[6] J. Choi and T. Y. Seo, *Identities involving series of the Riemann Zeta function*, Indian J. Pure Appl. Math. **30** (1999), 649–652.

[7] J. Choi and H. M. Srivastava, *Certain classes of series involving the Zeta function*, J. Math. Anal. Appl. **231** (1999), 91–117.

[8] J. Choi and H. M. Srivastava, *Certain classes of series associated with the Zeta function and multiple Gamma functions*, J. Comput. Appl. Math. **118** (2000), 87–109.

[9] J. Choi and H. M. Srivastava, *Explicit evaluation of Euler and related sums*, The Ramanujan J. **10** (2005), 51–70.

[10] H. M. Edwards, *Riemann's Zeta Function*, Dover Publications, Inc., Mineola, New York, 2001.

[11] G. H. Hardy, *Divergent Series*, Clarendon (Oxford University) Press, Oxford, London, and New York 1949.

[12] A. Ivić, *The Riemann Zeta-Function*, Dover Publications, Inc., Mineola, New York, 2003.

[13] K. Knopp, *Theory and Application of Infinite Series*, Second English Ed. (Translated from the Second German Ed. and Revised in accordance with the Fourth German Ed. by R. C. H. Young), Hafner Publishing Company, New York, 1951.

[14] Z. A. Melzak, *Companion to Concrete Mathematics*, Vol. **I**: Mathematical Techniques and Various Applications John Wiley and Sons, New York, London, Sydney, and Toronto, 1973.

[15] B. Riemann, *Über die Anzahl der Primzahlen unter einer gegebenen Grösse*, Monatsber. Akad. Berlin (1859), 671–680.

[16] J. D. Shallit and K. Zikan, *A theorem of Goldbach,* Amer. Math. Monthly **93** (1986), 402–403.

[17] O. Spiess, *Die Summe der reziproken Quadratzahlen*, in Festschrift zum 60 Geburtstag von Prof. Dr. Andreas Speiser (L. V. Ahlfors *et al.*, Editors), pp. 66-86, Füssli, Zürich, 1955.

[18] H. M. Srivastava, *A unified presentation of certain classes of series of the Riemann Zeta function*, Riv. Mat. Univ. Parma (4) **14** (1988), 1–23.

[19] H. M. Srivastava and J. Choi, *Series Associated with the Zeta and Related Functions*, Kluwer Acad. Publ., Dordrecht-Boston-London, 2001.

[20] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis* (4th ed.), Cambridge Univ. Press, Cambridge-London-New York, 1963.

[21] D. Zagier, *Values of Zeta functions and their applications*, First European Congress of Mathematics, Vol. II (Paris, 1992) (A. Joseph, F. Mignot, F. Murat, B. Prum, and R. Rentschler, Editors) pp. 497–512, Progress in Mathematics **120**, Birkhäuser, Basel, 1994.

*

Department of Mathematics
Dongguk University
Gyeongju 780-714, Republic of Korea
E-mail: junesang@mail.dongguk.ac.kr