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ON PRIME AND SEMIPRIME RINGS WITH SYMMETRIC n-DERIVATIONS

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Abstract. Let $n \geq 2$ be a fixed positive integer and let R be a noncommutative $n!$-torsion free semiprime ring. Suppose that there exists a symmetric n-derivation $\Delta : R^n \to R$ such that the trace of Δ is centralizing on R. Then the trace is commuting on R. If R is a $n!$-torsion free prime ring and $\Delta \neq 0$ under the same condition. Then R is commutative.

1. Introduction and preliminaries

Throughout this paper, R always represents an associative ring and Z is its center. Let $x, y, z \in R$. We write the notation $[y, x]$ for the commutator $yx - xy$ and make use of the identities $[xy, z] = [x, z]y + x[y, z]$ and $[x, yz] = [x, y]z + y[x, z]$. Recall that R is *semiprime* if $aRa = 0$ implies $a = 0$ and R is *prime* if $aRb = 0$ implies $a = 0$ or $b = 0$. A map $f : R \to R$ is said to be *commuting* on R if $[f(x), x] = 0$ holds for all $x \in R$. It is said that a map $f : R \to R$ is *centralizing* on R if $[f(x), x] \in Z$ is fulfilled for all $x \in R$. An additive map $D : R \to R$ is called a *derivation* if the Leibniz rule $D(xy) = D(x)y + xD(y)$ holds for all $x, y \in R$. Let $n \geq 2$ be a fixed positive integer and $R^n = R \times R \times \cdots \times R$. A map $\Delta : R^n \to R$ is said to be *symmetric* (or *permuting*) if the equation $\Delta(x_1, x_2, \cdots, x_n) = \Delta(x_{\pi(1)}, x_{\pi(2)}, \cdots, x_{\pi(n)})$ holds for all $x_i \in R$ and for every permutation $\{\pi(1), \pi(2), \cdots, \pi(n)\}$. Let us consider the following map:

Let $n \geq 2$ be a fixed positive integer. An n-additive map $\Delta : R^n \to R$ (i.e., additive in each argument) will be called an *n-derivation* if the

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relations

$$\begin{aligned}
\Delta(x_1x_1^{'}, x_2, \cdots, x_n) &= \Delta(x_1, x_2, \cdots, x_n)x_1^{'} + x_1\Delta(x_1^{'}, x_2, \cdots, x_n),\\
\Delta(x_1, x_2x_2^{'}, \cdots, x_n) &= \Delta(x_1, x_2, \cdots, x_n)x_2^{'} + x_2\Delta(x_1, x_2^{'}, \cdots, x_n),\\
&\vdots\\
\Delta(x_1, x_2, \cdots, x_nx_n^{'}) &= \Delta(x_1, x_2, \cdots, x_n)x_n^{'} + x_n\Delta(x_1, x_2, \cdots, x_n^{'})
\end{aligned}$$

are valid for all $x_i, x_i^{'} \in R$. Of course, an 1-derivation is a derivation and a 2-derivation is called a bi-derivation. If Δ is symmetric, then the above equalities are equivalent to each other. In particular, in case when $n = 2$, namely, Δ is a symmetric bi-derivation on noncommutative 2-torsion free prime rings, M. Brešar [1, Theorem 3.5] proved that $\Delta = 0$.

Let $n \geq 2$ be a fixed positive integer. If R is commutative, then a map $\Delta : R^n \to R$ defined by

$$(x_1, x_2, \cdots, x_n) \mapsto D(x_1)D(x_2)\cdots D(x_n)$$

for all $x_i \in R$, $i = 1, 2, \cdots, n$ is a symmetric n-derivation, where D is a derivation on R.

On the other hand, let

$$R = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \,\middle|\, a, b \in \mathbb{C} \right\},$$

where $\mathbb{C}$ is a complex field. It is clear that R is a noncommutative ring under matrix addition and matrix multiplication. We define a map $\Delta : R^n \to R$ by

$$\left(\begin{pmatrix} a_1 & b_1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ 0 & 0 \end{pmatrix}, \cdots, \begin{pmatrix} a_n & b_n \\ 0 & 0 \end{pmatrix} \right) \mapsto \begin{pmatrix} 0 & a_1a_2\cdots a_n \\ 0 & 0 \end{pmatrix}.$$

Then it is easy to see that Δ is a symmetric n-derivation.

Let $n \geq 2$ be a fixed positive integer and let a map $\delta : R \to R$ defined by $\delta(x) = \Delta(x_1, x_2, \cdots x_n)$ for all $x \in R$, where $\Delta : R^n \to R$ is a symmetric map, be the *trace* of Δ. It is obvious that, in case when $\Delta : R^n \to R$ is a symmetric map which is also n-additive, the trace δ of Δ satisfies the relation

$$\delta(x + y) = \delta(x) + \delta(y) + \sum_{k=1}^{n-1} {}_n\mathrm{C}_k\, h_k(x; y)$$

for all $x, y \in R$, where ${}_n\mathrm{C}_k = \binom{n}{k}$ and

$$h_k(x;y) = \Delta(\overbrace{x, x, \cdots, x}^{n-k \text{ times}}, \overbrace{y, y, \cdots, y}^{k \text{ times}}).$$

Since we have

$$\begin{aligned}\Delta(0, x_2, \cdots, x_n) &= \Delta(0+0, x_2, \cdots, x_n) \\ &= \Delta(0, x_2, \cdots, x_n) + \Delta(0, x_2, \cdots, x_n)\end{aligned}$$

for all $x_i \in R$, $i = 2, 3, \cdots, n$, we obtain $\Delta(0, x_2, \cdots, x_n) = 0$ for all $x_i \in R$, $i = 2, 3, \cdots, n$. Hence we get

$$\begin{aligned}0 &= \Delta(0, x_2, \cdots, x_n) \\ &= \Delta(x_1 - x_1, x_2, \cdots, x_n) \\ &= \Delta(x_1, x_2, \cdots, x_n) + \Delta(-x_1, x_2, \cdots, x_n)\end{aligned}$$

and so we see that $\Delta(-x_1, x_2, \cdots, x_n) = -\Delta(x_1, x_2, \cdots, x_n)$ for all $x_i \in R$, $i = 1, 2, \cdots, n$. This tells us that δ is an odd function if n is odd and δ is an even function if n is even.

A study on the theory of centralizing (commuting) maps on prime rings was initiated by the classical result of E.C. Posner [5] which states that the existence of a nonzero centralizing derivation on a prime ring R implies that R is commutative. Since then, a great deal of work in this context has been done by a number of authors (see, e.g., [1] and references therein). For example, as a study concerning centralizing (commuting) maps, J. Vukman [6, 7] investigated symmetric bi-derivations on prime and semiprime rings. In [3], we obtained the similar results to Posner's and Vukman's ones for permuting 3-derivations on prime and semiprime rings. Our main purpose in this paper is to apply the result due to E.C. Posner [5, Theorem 2] to symmetric n-derivations.

2. Results

We first precede the proof of our results by two well-known lemmas.

LEMMA 2.1 ([4]). *Let R be a prime ring. Let $D : R \to R$ be a derivation and $a \in R$. If $aD(x) = 0$ holds for all $x \in R$, then we have either $a = 0$ or $D = 0$.*

LEMMA 2.2 ([2]). *Let n be a fixed positive integer and let R be a $n!$-torsion free ring. Suppose that $y_1, y_2, \cdots, y_n \in R$ satisfy $\lambda y_1 + \lambda^2 y_2 + \cdots + \lambda^n y_n = 0$ for $\lambda = 1, 2, \cdots, n$. Then $y_i = 0$ for all i.*

We start our investigation of symmetric n-derivations with the following result.

Theorem 2.3. *Let $n \geq 2$ be a fixed positive integer and let R be a noncommutative $n!$-torsion free prime ring. Suppose that there exists a symmetric n-derivation $\Delta : R^n \to R$ such that the trace δ of Δ is commuting on R. Then we have $\Delta = 0$.*

Proof. Suppose that

$$[\delta(x), x] = 0 \tag{1}$$

for all $x \in R$. Let λ $(1 \leq \lambda \leq n)$ be any integer. Substituting $x + \lambda y$ for x in (1) and using (1), we get

$$\begin{aligned} 0 &= \lambda\big\{[\delta(x), y] + {}_n\mathrm{C}_1[h_1(x;y), x]\big\} \\ &\quad + \lambda^2\big\{{}_n\mathrm{C}_1[h_1(x;y), y] + {}_n\mathrm{C}_2[h_2(x;y), x]\big\} \\ &\quad + \cdots + \lambda^n\big\{[\delta(y), x] + {}_n\mathrm{C}_{n-1}[h_{n-1}(x;y), y]\big\} \end{aligned} \tag{2}$$

for all $x, y \in R$. From Lemma 2.2 and (2), we infer that

$$[\delta(x), y] + n[h_1(x;y), x] = 0 \tag{3}$$

for all $x, y \in R$.

Let us write in (3) xy instead of y. Then we have

$$\begin{aligned} 0 &= [\delta(x), xy] + n[h_1(x;xy), x] \\ &= x\big\{[\delta(x), y] + n[h_1(x;y), x]\big\} + n\delta(x)[y, x] \end{aligned}$$

which implies that

$$n\delta(x)[y, x] = 0 = \delta(x)[y, x] \tag{4}$$

for all $x, y \in R$. From (4) and Lemma 2.1, it follows that

$$\delta(x) = 0$$

for all $x \in R$ $(x \notin Z)$ since for any fixed $x \in R$, a map $y \mapsto [y, x]$ is a derivation on R.

Now, let $x \in R$ $(x \in Z)$ and $y \in R$ $(y \notin Z)$. Then $y + \lambda x \notin Z$. Thus we obtain

$$\begin{aligned} 0 = \delta(y + \lambda x) &= \delta(y) + \lambda^n \delta(x) + \sum_{k=1}^{n-1} \lambda^k {}_n\mathrm{C}_k\, h_k(y;x) \\ &= \sum_{k=1}^{n-1} \lambda^k {}_n\mathrm{C}_k\, h_k(y;x) + \lambda^n \delta(x) \end{aligned}$$

for all $x, y \in R$ and applying this relation to Lemma 2.2 yields

$$\delta(x) = 0$$

for all $x \in R$ $(x \in Z)$. Therefore, we conclude that

$$\delta(x) = 0 \tag{5}$$

for all $x \in R$.

For each $k = 1, 2, \cdots, n$, let

$$P_k(x) = \Delta(\overbrace{x, x, \cdots, x}^{k \text{ times}}, x_{k+1}, x_{k+2}, \cdots, x_n),$$

where $x, x_i \in R$, $i = k+1, k+2, \cdots, n$. Let μ $(1 \le \mu \le n-1)$ be any integer. By (5), the relation

$$\begin{aligned} 0 = \delta(\mu x + x_n) &= P_n(\mu x + x_n) \\ &= \mu^n \delta(x) + \delta(x_n) + \sum_{k=1}^{n-1} \mu^k {}_n\mathrm{C}_k \, P_k(x) \\ &= \sum_{k=1}^{n-1} \mu^k {}_n\mathrm{C}_k \, P_k(x) \end{aligned}$$

is true for all $x, x_n \in R$, that is,

$$\sum_{k=1}^{n-1} \mu^k {}_n\mathrm{C}_k \, P_k(x) = 0 \tag{6}$$

for all $x \in R$. Thus Lemma 2.1 and (6) give

$${}_n\mathrm{C}_{n-1} \, P_{n-1}(x) = 0 = P_{n-1}(x) \tag{7}$$

for all $x \in R$. Let ν $(1 \le \nu \le n-2)$ be any integer. By (7), the relation

$$0 = P_{n-1}(\nu x + x_{n-1}) = \nu^{n-1} P_{n-1}(x) + P_{n-1}(x_{n-1}) + \sum_{k=1}^{n-2} \nu^k {}_n\mathrm{C}_k \, P_k(x)$$

holds for all $x, x_{n-1} \in R$ and hence we see that

$$\sum_{k=1}^{n-2} \nu^k {}_n\mathrm{C}_k \, P_k(x) = 0 \tag{8}$$

for all $x \in R$. Using Lemma 2.1 and (8), we get

$${}_n\mathrm{C}_{n-2} \, P_{n-2}(x) = 0 = P_{n-2}(x)$$

for all $x \in R$. Now if we continue to carry out the same method as above, we finally arrive at

$$ {}_n\mathrm{C}_1\, P_1(x) = 0 = P_1(x) $$

for all $x \in R$ which means

$$ \Delta(x_1, x_2, \cdots, x_n) = 0 $$

for all $x_i \in R$. The proof of the theorem is complete. □

Here we need the following lemma.

LEMMA 2.4. *Let n be a fixed positive integer and let R be a $n!$-torsion free ring. Suppose that $y_1, y_2, \cdots, y_n \in R$ satisfy $\lambda y_1 + \lambda^2 y_2 + \cdots + \lambda^n y_n \in Z$ for $\lambda = 1, 2, \cdots, n$. Then $y_i \in Z$ for all i.*

Proof. The arguments used in the proof of Lemma 2.2 carry over almost verbatim. □

We continue with the next result for symmetric n-derivations on semiprime rings.

THEOREM 2.5. *Let $n \geq 2$ be a fixed positive integer and let R be a noncommutative $n!$-torsion free semiprime ring. Suppose that there exists a symmetric n-derivation $\Delta : R^n \to R$ such that the trace δ of Δ is centralizing on R. Then δ is commuting on R.*

Proof. Assume that

$$ [\delta(x), x] \in Z \tag{9} $$

for all $x \in R$. Let λ $(1 \leq \lambda \leq n)$ be any positive integer. By replacing x by $x + \lambda y$ in (9) and utilizing (9), we obtain

$$ \begin{aligned} Z \ni\ & \lambda\big\{[\delta(x), y] + {}_n\mathrm{C}_1[h_1(x;y), x]\big\} \\ & + \lambda^2\big\{{}_n\mathrm{C}_1[h_1(x;y), y] + {}_n\mathrm{C}_2[h_2(x;y), x]\big\} \\ & + \cdots + \lambda^n\big\{[\delta(y), x] + {}_n\mathrm{C}_{n-1}[h_{n-1}(x;y), y]\big\} \end{aligned} \tag{10} $$

for all $x, y \in R$. From Lemma 2.4 and (10), it follows that

$$ [\delta(x), y] + n[h_1(x;y), x] \in Z \tag{11} $$

for all $x, y \in R$. Taking $y = x^2$ in (11) and invoking (11) show

$$ Z \ni [\delta(x), x^2] + n[h_1(x;x^2), x] = (2n+2)[\delta(x), x]x \tag{12} $$

for all $x \in R$ and commuting with $\delta(x)$ in (12) gives

$$ (2n+2)[\delta(x), x]^2 = 0 \tag{13} $$

for all $x \in R$.

On the other hand, substituting y by xy in (11), we obtain

$$\begin{aligned} Z \ni & [\delta(x), xy] + n[h_1(x; xy), x] \\ & = x\big\{[\delta(x), y] + n[h_1(x; y), x]\big\} + n\delta(x)[y, x] + (n+1)[\delta(x), x]y \end{aligned}$$

for all $x, y \in R$ and so we have

$$\begin{aligned} & [x\big\{[\delta(x), y] + n[h_1(x; y), x]\big\}, x] \\ & + [n\delta(x)[y, x] + (n+1)[\delta(x), x]y, x] = 0 \end{aligned} \tag{14}$$

for all $x, y \in R$. Using (11), it follows from (14) that

$$n\delta(x)[[y, x], x] + (2n+1)[\delta(x), x][y, x] = 0 \tag{15}$$

for all $x, y \in R$.

The substitution $\delta(x)y$ for y in (15) and the relation (9) yield

$$\begin{aligned} 0 = & \delta(x)\big\{n\delta(x)[[y, x], x] + (2n+1)[\delta(x), x][y, x]\big\} \\ & + 2n\delta(x)[\delta(x), x][y, x] + (2n+1)[\delta(x), x]^2 y \end{aligned}$$

for all $x, y \in R$ which, according to (15), reduces to

$$2n\delta(x)[\delta(x), x][y, x] + (2n+1)[\delta(x), x]^2 y = 0 \tag{16}$$

for all $x, y \in R$. Taking $y = [\delta(x), x]$ into (16), we arrive at $(2n+1)[\delta(x), x]^3 = 0$ and so we have

$$(2n+1)[\delta(x), x]^2 R\, (2n+1)[\delta(x), x]^2 = 0$$

for all $x \in R$. From the semiprimeness of R, we see that

$$(2n+1)[\delta(x), x]^2 = 0 \tag{17}$$

for all $x \in R$. Now, combining (17) with (13) leads to the relation $[\delta(x), x]^2 = 0$ for all $x \in R$. Since the center of a semiprime ring contains no nonzero nilpotent elements, we conclude that $[\delta(x), x] = 0$ for all $x \in R$. This completes the proof of the theorem. □

Our main result, which is an analogue of Posner's theorem [5, Theorem 2], is as follows:

THEOREM 2.6. *Let $n \geq 2$ be a fixed positive integer and let R be a $n!$-torsion free prime ring. Suppose that there exists a nonzero symmetric n-derivation $\Delta : R^n \to R$ such that the trace δ of Δ is centralizing on R. Then R is commutative.*

Proof. Suppose that R is noncommutative. Then it follows from Theorem 2.5 that δ is commuting on R. Hence Theorem 2.3 gives $\Delta = 0$ which is a contradiction. This guarantees the conclusion of the theorem. □

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