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BIPOLAR FUZZY TRANSLATIONS IN BCK/BCI-ALGEBRAS

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Abstract. A bipolar fuzzy translation and a bipolar fuzzy S-extension of a bipolar fuzzy subalgebra in a BCK/BCI-algebra are introduced, and related properties are investigated.

1. Introduction

Fuzzy sets are a kind of useful mathematical structure to represent a collection of objects whose boundary is vague. There are several kinds of fuzzy set extensions in the fuzzy set theory, for example, intuitionistic fuzzy sets, interval-valued fuzzy sets, vague sets etc. Bipolar-valued fuzzy sets, which are introduced by Lee ([4], [5]), are an extension of fuzzy sets whose membership degree range is enlarged from the interval $[0, 1]$ to $[-1, 1]$. Using the notion of bipolar-valued fuzzy sets, Jun et al. [2] dealt with subalgebras and closed ideals of BCH-algebras based on bipolar-valued fuzzy sets. Lee [3] discussed bipolar fuzzy subalgebras and bipolar fuzzy ideals in BCK/BCI-algebras. In this paper, we introduce the notion of a bipolar fuzzy translation and a bipolar fuzzy S-extension of a bipolar fuzzy subalgebra in a BCK/BCI-algebra, and investigate its properties.

2. Preliminaries

An algebra $(X; *, \theta)$ of type $(2, 0)$ is called a *BCI-algebra* if it satisfies the following conditions:

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(I) $(\forall x, y, z \in X)\ (((x * y) * (x * z)) * (z * y) = \theta)$,
(II) $(\forall x, y \in X)\ ((x * (x * y)) * y = \theta)$,
(III) $(\forall x \in X)\ (x * x = \theta)$,
(IV) $(\forall x, y \in X)\ (x * y = \theta, y * x = \theta \Rightarrow x = y)$.

We can define a partial order '$\leq$' on X by $x \leq y$ if and only if $x * y = \theta$. Any BCI-algebra X has the following properties:

(a1) $(\forall x \in X)\ (x * \theta = x)$.
(a2) $(\forall x, y, z \in X)\ ((x * y) * z = (x * z) * y)$.
(a3) $(\forall x, y, z \in X)\ (x \leq y \Rightarrow x * z \leq y * z,\ z * y \leq z * x)$.
(a4) $(\forall x \in X)\ (x \leq \theta \Rightarrow x = \theta)$.

A subset S of a BCK/BCI-algebra X is called a *subalgebra* of X if $x * y \in S$ for all $x, y \in S$.

We refer the reader to the books [1] and [6] for further information regarding BCK/BCI-algebras.

Let X be the universe of discourse. A *bipolar-valued fuzzy set* φ in X is an object having the form

$$\varphi = \{(x, \varphi^-(x), \varphi^+(x)) \mid x \in X\}$$

where $\varphi^- : X \to [-1, 0]$ and $\varphi^+ : X \to [0, 1]$ are mappings. The positive membership degree $\varphi^+(x)$ denoted the satisfaction degree of an element x to the property corresponding to a bipolar-valued fuzzy set $\varphi = \{(x, \varphi^-(x), \varphi^+(x)) \mid x \in X\}$, and the negative membership degree $\varphi^-(x)$ denotes the satisfaction degree of x to some implicit counter-property of $\varphi = \{(x, \varphi^-(x), \varphi^+(x)) \mid x \in X\}$. If $\varphi^+(x) \neq 0$ and $\varphi^-(x) = 0$, it is the situation that x is regarded as having only positive satisfaction for $\varphi = \{(x, \varphi^-(x), \varphi^+(x)) \mid x \in X\}$. If $\varphi^+(x) = 0$ and $\varphi^-(x) \neq 0$, it is the situation that x does not satisfy the property of $\varphi = \{(x, \varphi^-(x), \varphi^+(x)) \mid x \in X\}$ but somewhat satisfies the counter-property of $\varphi = \{(x, \varphi^-(x), \varphi^+(x)) \mid x \in X\}$. It is possible for an element x to be $\varphi^+(x) \neq 0$ and $\varphi^-(x) \neq 0$ when the membership function of the property overlaps that of its counter-property over some portion of the domain (see [5]). For the sake of simplicity, we shall use the symbol $\varphi = (X; \varphi^-, \varphi^+)$ for the bipolar-valued fuzzy set $\varphi = \{(x, \varphi^-(x), \varphi^+(x)) \mid x \in X\}$, and use the notion of bipolar fuzzy sets instead of the notion of bipolar-valued fuzzy sets.

For two bipolar fuzzy sets $\varphi = (X; \varphi^-, \varphi^+)$ and $\psi = (X; \psi^-, \psi^+)$ in X, the *union* of $\varphi = (X; \varphi^-, \varphi^+)$ and $\psi = (X; \psi^-, \psi^+)$ is a bipolar fuzzy set $\varphi \cup \psi = (X; (\varphi \cup \psi)^-, (\varphi \cup \psi)^+)$ where

$$(\varphi \cup \psi)^- : X \to [-1, 0],\ x \mapsto \min\{\varphi^-(x), \psi^-(x)\},$$

$$(\varphi \cup \psi)^+ : X \to [0,1],\ x \mapsto \max\{\varphi^+(x), \psi^+(x)\}.$$

The *intersection* of $\varphi = (X; \varphi^-, \varphi^+)$ and $\psi = (X; \psi^-, \psi^+)$ is a bipolar fuzzy set $\varphi \cap \psi = (X; (\varphi \cap \psi)^-, (\varphi \cap \psi)^+)$ where

$$(\varphi \cap \psi)^- : X \to [-1,0],\ x \mapsto \max\{\varphi^-(x), \psi^-(x)\},$$

$$(\varphi \cap \psi)^+ : X \to [0,1],\ x \mapsto \min\{\varphi^+(x), \psi^+(x)\}.$$

3. Bipolar fuzzy translations

In what follows, let X denote a BCK/BCI-algebra unless otherwise specified.

Definition 3.1. [3] A bipolar fuzzy set $\varphi = (X; \varphi^-, \varphi^+)$ in X is called a *bipolar fuzzy subalgebra* of X if it satisfies:

$$\begin{aligned} \varphi^-(x * y) &\le \max\{\varphi^-(x), \varphi^-(y)\}, \\ \varphi^+(x * y) &\ge \min\{\varphi^+(x), \varphi^+(y)\} \end{aligned} \tag{3.1}$$

for all $x, y \in X$.

For any bipolar fuzzy set $\varphi = (X; \varphi^-, \varphi^+)$ in X, we denote

$$\bot := -1 - \inf\{\varphi^-(x) \mid x \in X\},$$

$$\top := 1 - \sup\{\varphi^+(x) \mid x \in X\}.$$

Definition 3.2. Let $\varphi = (X; \varphi^-, \varphi^+)$ be a bipolar fuzzy set in X and $(\alpha, \beta) \in [\bot, 0] \times [0, \top]$. By a *bipolar fuzzy* (α, β)*-translation* of $\varphi = (X; \varphi^-, \varphi^+)$ we mean a bipolar fuzzy set $\varphi^T_{(\alpha,\beta)} = \big(X; \varphi^-_{(\alpha,T)}, \varphi^+_{(\beta,T)}\big)$ where

$$\varphi^-_{(\alpha,T)} : X \to [-1,0],\ x \mapsto \varphi^-(x) + \alpha,$$

$$\varphi^+_{(\beta,T)} : X \to [0,1],\ x \mapsto \varphi^+(x) + \beta.$$

Theorem 3.3. *Let $\varphi = (X; \varphi^-, \varphi^+)$ be a bipolar fuzzy subalgebra of X and $(\alpha, \beta) \in [\bot, 0] \times [0, \top]$. Then the bipolar fuzzy (α, β)-translation $\varphi^T_{(\alpha,\beta)} = \big(X; \varphi^-_{(\alpha,T)}, \varphi^+_{(\beta,T)}\big)$ of $\varphi = (X; \varphi^-, \varphi^+)$ is a bipolar fuzzy subalgebra of X.*

Proof. Let $x, y \in X$. Then

$$\begin{aligned} \varphi^-_{(\alpha,T)}(x * y) &= \varphi^-(x * y) + \alpha \ \le\ \max\{\varphi^-(x), \varphi^-(y)\} + \alpha \\ &= \max\{\varphi^-(x) + \alpha, \varphi^-(y) + \alpha\} \\ &= \max\{\varphi^-_{(\alpha,T)}(x), \varphi^-_{(\alpha,T)}(y)\} \end{aligned}$$

and

$$\begin{aligned}\varphi^+_{(\beta,T)}(x*y) &= \varphi^+(x*y)+\beta \ \geq\ \min\{\varphi^+(x),\varphi^+(y)\}+\beta\\ &= \min\{\varphi^+(x)+\beta,\varphi^+(y)+\beta\}\\ &= \min\{\varphi^+_{(\beta,T)}(x),\varphi^+_{(\beta,T)}(y)\}\end{aligned}$$

Hence $\varphi^T_{(\alpha,\beta)} = \left(X;\varphi^-_{(\alpha,T)},\varphi^+_{(\beta,T)}\right)$ is a bipolar fuzzy subalgebra of X. □

THEOREM 3.4. *Let $\varphi = (X;\varphi^-,\varphi^+)$ be a bipolar fuzzy set in X such that the bipolar fuzzy (α,β)-translation $\varphi^T_{(\alpha,\beta)} = \left(X;\varphi^-_{(\alpha,T)},\varphi^+_{(\beta,T)}\right)$ of $\varphi = (X;\varphi^-,\varphi^+)$ is a fuzzy subalgebra of X for some $(\alpha,\beta)\in[\bot,0]\times[0,\top]$. Then $\varphi = (X;\varphi^-,\varphi^+)$ is a bipolar fuzzy subalgebra of X.*

Proof. Assume that $\varphi^T_{(\alpha,\beta)} = \left(X;\varphi^-_{(\alpha,T)},\varphi^+_{(\beta,T)}\right)$ is a bipolar fuzzy subalgebra of X for some $(\alpha,\beta)\in[\bot,0]\times[0,\top]$. For any $x,y\in X$, we have

$$\begin{aligned}&\varphi^-(x*y)+\alpha = \varphi^-_{(\alpha,T)}(x*y) \ \leq\ \max\{\varphi^-_{(\alpha,T)}(x),\varphi^-_{(\alpha,T)}(y)\}\\ &= \max\{\varphi^-(x)+\alpha,\varphi^-(y)+\alpha\} = \max\{\varphi^-(x),\varphi^-(y)\}+\alpha\end{aligned}$$

and

$$\begin{aligned}&\varphi^+(x*y)+\beta = \varphi^+_{(\beta,T)}(x*y) \ \geq\ \min\{\varphi^+_{(\beta,T)}(x),\varphi^+_{(\beta,T)}(y)\}\\ &= \min\{\varphi^+(x)+\beta,\varphi^+(y)+\beta\} = \min\{\varphi^+(x),\varphi^+(y)\}+\beta\end{aligned}$$

which imply that $\varphi^-(x*y)\leq\max\{\varphi^-(x),\varphi^-(y)\}$ and $\varphi^+(x*y)\geq\min\{\varphi^+(x),\varphi^+(y)\}$ respectively for all $x,y\in X$. Hence $\varphi = (X;\varphi^-,\varphi^+)$ is a fuzzy subalgebra of X. □

DEFINITION 3.5. Let $\varphi = (X;\varphi^-,\varphi^+)$ and $\psi = (X;\psi^-,\psi^+)$ be bipolar fuzzy sets in X. If $\varphi^-(x)\geq\psi^-(x)$ and $\varphi^+(x)\leq\psi^+(x)$ for all $x\in X$, then we say that $\psi = (X;\psi^-,\psi^+)$ is a *bipolar fuzzy extension* of $\varphi = (X;\varphi^-,\varphi^+)$.

DEFINITION 3.6. Let $\varphi = (X;\varphi^-,\varphi^+)$ and $\psi = (X;\psi^-,\psi^+)$ be bipolar fuzzy sets in X. Then $\psi = (X;\psi^-,\psi^+)$ is called a *bipolar fuzzy S-extension* of $\varphi = (X;\varphi^-,\varphi^+)$ if the following assertions are valid:

(i) $\psi = (X;\psi^-,\psi^+)$ is a bipolar fuzzy extension of $\varphi = (X;\varphi^-,\varphi^+)$.
(ii) If $\varphi = (X;\varphi^-,\varphi^+)$ is a bipolar fuzzy subalgebra of X, then $\psi = (X;\psi^-,\psi^+)$ is a bipolar fuzzy subalgebra of X.

By means of the definition of a bipolar fuzzy (α,β)-translation, we know that $\varphi^-_{(\alpha,T)}(x)\leq\varphi^-(x)$ and $\varphi^+_{(\beta,T)}(x)\geq\varphi^+(x)$ for all $x\in X$. Hence we have the following theorem.

$*$	θ	a	b	c	d
θ	θ	θ	θ	θ	θ
a	a	θ	a	θ	θ
b	b	b	θ	b	θ
c	c	a	c	θ	a
d	d	d	d	d	θ

TABLE 1. $*$-multiplication table for X

THEOREM 3.7. *Let $\varphi = (X; \varphi^-, \varphi^+)$ be a bipolar fuzzy subalgebra of X and $(\alpha, \beta) \in [\bot, 0] \times [0, \top]$. Then the bipolar fuzzy (α, β)-translation $\varphi^T_{(\alpha,\beta)} = \big(X; \varphi^-_{(\alpha,T)}, \varphi^+_{(\beta,T)}\big)$ of $\varphi = (X; \varphi^-, \varphi^+)$ is a bipolar fuzzy S-extension of $\varphi = (X; \varphi^-, \varphi^+)$.*

The converse of Theorem 3.7 is not true in general as seen in the following example.

EXAMPLE 3.8. *Consider a BCK-algebra $X = \{\theta, a, b, c, d\}$ where the $*$-multiplication is defined by Table 1. Let $\varphi = (X; \varphi^-, \varphi^+)$ be a bipolar fuzzy set in X defined by*

X	θ	a	b	c	d
φ^-	-0.6	-0.4	-0.3	-0.5	-0.1
φ^+	0.8	0.5	0.3	0.6	0.2

Then $\varphi = (X; \varphi^-, \varphi^+)$ is a bipolar fuzzy subalgebra of X. Let $\psi = (X; \psi^-, \psi^+)$ be a bipolar fuzzy set in X given by

X	θ	a	b	c	d
ψ^-	-0.63	-0.42	-0.37	-0.54	-0.16
ψ^+	0.84	0.56	0.38	0.67	0.21

Then $\psi = (X; \psi^-, \psi^+)$ is a bipolar fuzzy S-extension of $\varphi = (X; \varphi^-, \varphi^+)$. But it is not the bipolar fuzzy (α, β)-translation $\varphi^T_{(\alpha,\beta)} = \big(X; \varphi^-_{(\alpha,T)}, \varphi^+_{(\beta,T)}\big)$ of $\varphi = (X; \varphi^-, \varphi^+)$ for all $(\alpha, \beta) \in [\bot, 0] \times [0, \top]$.

Clearly, the intersection of bipolar fuzzy S-extensions of a bipolar fuzzy subalgebra $\varphi = (X; \varphi^-, \varphi^+)$ is a bipolar fuzzy S-extension of $\varphi =$

$*$	θ	a	b	c	d
θ	θ	θ	θ	θ	θ
a	a	θ	θ	θ	θ
b	b	a	θ	θ	θ
c	c	a	a	θ	θ
d	d	c	c	a	θ

TABLE 2. $*$-multiplication table for X

$(X;\varphi^-,\varphi^+)$. But the union of bipolar fuzzy S-extensions of a bipolar fuzzy subalgebra $\varphi = (X;\varphi^-,\varphi^+)$ is not a bipolar fuzzy S-extension of $\varphi = (X;\varphi^-,\varphi^+)$ as seen in the following example.

EXAMPLE 3.9. *Consider a BCK-algebra $X = \{\theta, a, b, c, d\}$ where the $*$-multiplication is defined by Table 2. Let $\varphi = (X;\varphi^-,\varphi^+)$ be a bipolar fuzzy set in X defined by*

X	θ	a	b	c	d
φ^-	-0.6	-0.3	-0.5	-0.2	-0.1
φ^+	0.7	0.4	0.6	0.3	0.3

Then $\varphi = (X;\varphi^-,\varphi^+)$ is a bipolar fuzzy subalgebra of X. Let $\psi = (X;\psi^-,\psi^+)$ and $\kappa = (X;\kappa^-,\kappa^+)$ be bipolar fuzzy sets in X given by

X	θ	a	b	c	d
ψ^-	-0.66	-0.33	-0.55	-0.22	-0.22
ψ^+	0.8	0.6	0.8	0.4	0.4

and

X	θ	a	b	c	d
κ^-	-0.7	-0.4	-0.6	-0.4	-0.3
κ^+	0.9	0.6	0.6	0.6	0.7

respectively. Then $\psi = (X;\psi^-,\psi^+)$ and $\kappa = (X;\kappa^-,\kappa^+)$ are bipolar fuzzy S-extensions of $\varphi = (X;\varphi^-,\varphi^+)$. But the union $\psi \cup \kappa$ is not a *bipolar fuzzy S-extension of* $\varphi = (X; \varphi^-, \varphi^+)$ *since*

$$\begin{aligned}(\psi \cup \kappa)^-(d * b) &= \min\{\psi^-(d*b), \kappa^-(d*b)\} = -0.4 \\ &\leq -0.3 = \max\{(\psi \cup \kappa)^-(d), (\psi \cup \kappa)^-(b)\},\end{aligned}$$

but

$$\begin{aligned}(\psi \cup \kappa)^+(d * b) &= \max\{\psi^+(d*b), \kappa^+(d*b)\} = 0.6 \\ &\ngeq 0.7 = \min\{(\psi \cup \kappa)^+(d), (\psi \cup \kappa)^+(b)\}.\end{aligned}$$

For a bipolar fuzzy set $\varphi = (X; \varphi^-, \varphi^+)$ in X, consider the following two sets:

$$N_\alpha(\varphi^-; t^-) := \{x \in X \mid \varphi^-(x) \leq t^- - \alpha\},$$
$$P_\beta(\varphi^+; t^+) := \{x \in X \mid \varphi^+(x) \geq t^+ - \beta\}$$

where $(\alpha, \beta) \in [\bot, 0] \times [0, \top]$ and $(t^-, t^+) \in [-1, \alpha] \times [\beta, 1]$. If $\varphi = (X; \varphi^-, \varphi^+)$ is a bipolar fuzzy subalgebra of X, then it is cleat that $N_\alpha(\varphi^-; t^-)$ and $P_\beta(\varphi^+; t^+)$ are subalgebras of X for all $(t^-, t^+) \in \mathrm{Im}(\varphi^-) \times \mathrm{Im}(\varphi^+)$ with $t^- \leq \alpha$ and $t^+ \geq \beta$. However, if we do not give a condition that $\varphi = (X; \varphi^-, \varphi^+)$ is a bipolar fuzzy subalgebra of X then at least one of $N_\alpha(\varphi^-; t^-)$ and $P_\beta(\varphi^+; t^+)$ may not be a subalgebra of X as seen in the following example.

EXAMPLE 3.10. *Let* $X = \{\theta, a, b, c, d\}$ *be a BCK-algebra which is given in Example 3.9. Define a bipolar fuzzy set* $\varphi = (X; \varphi^-, \varphi^+)$ *in* X *by*

X	θ	a	b	c	d
φ^-	-0.8	-0.4	-0.7	-0.2	-0.2
φ^+	0.7	0.4	0.6	0.3	0.5

Then $\varphi = (X; \varphi^-, \varphi^+)$ *is not a bipolar fuzzy subalgebra of* X *since*

$$\varphi^+(d * b) = 0.3 \ngeq 0.5 = \min\{\varphi^+(d), \varphi^+(b)\}.$$

Take $(\alpha, \beta) = (-0.15, 0.1)$ *and* $(t^-, t^+) = (-0.5, 0.5)$. *Then* $N_\alpha(\varphi^-; t^-) = \{\theta, a, b\}$ *is a subalgebra of* X, *but* $P_\beta(\varphi^+; t^+) = \{\theta, a, b, d\}$ *is not a subalgebra of* X *since* $d * b = c \notin P_\beta(\varphi^+; t^+)$.

THEOREM 3.11. *Let* $\varphi = (X; \varphi^-, \varphi^+)$ *be a bipolar fuzzy set in* X *and* $(\alpha, \beta) \in [\bot, 0] \times [0, \top]$. *Then the bipolar fuzzy* (α, β)*-translation* $\varphi^T_{(\alpha,\beta)} = \left(X; \varphi^-_{(\alpha,T)}, \varphi^+_{(\beta,T)}\right)$ *of* $\varphi = (X; \varphi^-, \varphi^+)$ *is a bipolar fuzzy subalgebra of* X *if and only if* $N_\alpha(\varphi^-; t^-)$ *and* $P_\beta(\varphi^+; t^+)$ *are subalgebras of* X *for all* $(t^-, t^+) \in \mathrm{Im}(\varphi^-) \times \mathrm{Im}(\varphi^+)$ *with* $t^- \leq \alpha$ *and* $t^+ \geq \beta$.

Proof. Necessity is clear. To prove the sufficiency, assume that there exist $a, b, c, d \in X$ such that $\varphi^-_{(\alpha,T)}(a * b) > \max\{\varphi^-_{(\alpha,T)}(a), \varphi^-_{(\alpha,T)}(b)\}$ and $\varphi^+_{(\beta,T)}(c * d) < \min\{\varphi^+_{(\beta,T)}(c), \varphi^+_{(\beta,T)}(d)\}$. Then

$$\varphi^-_{(\alpha,T)}(a * b) > k^- \geq \max\{\varphi^-_{(\alpha,T)}(a), \varphi^-_{(\alpha,T)}(b)\}$$

and

$$\varphi^+_{(\beta,T)}(c * d) < k^+ \leq \min\{\varphi^+_{(\beta,T)}(c), \varphi^+_{(\beta,T)}(d)\}$$

for some $(k^-, k^+) \in [-1, \alpha) \times (\beta, 1]$. It follows that $\varphi^-(a) \leq k^- - \alpha$, $\varphi^-(b) \leq k^- - \alpha$, $\varphi^+(c) \geq k^+ - \beta$ and $\varphi^+(d) \geq k^+ - \beta$, but $\varphi^-(a * b) > k^- - \alpha$ and $\varphi^+(c * d) < k^+ - \beta$. This shows that $a, b \in N_\alpha(\varphi^-; k^-)$ but $a * b \notin N_\alpha(\varphi^-; k^-)$, and $c, d \in P_\beta(\varphi^+; k^+)$ but $c * d \notin P_\beta(\varphi^+; k^+)$. This is a contradiction. Therefore $\varphi^-_{(\alpha,T)}(x * y) \leq \max\{\varphi^-_{(\alpha,T)}(x), \varphi^-_{(\alpha,T)}(y)\}$ and $\varphi^+_{(\beta,T)}(x * y) \geq \min\{\varphi^+_{(\beta,T)}(x), \varphi^+_{(\beta,T)}(y)\}$ for all $x, y \in X$. Hence $\varphi^T_{(\alpha,\beta)} = \left(X; \varphi^-_{(\alpha,T)}, \varphi^+_{(\beta,T)}\right)$ is a bipolar fuzzy subalgebra of X. □

THEOREM 3.12. *Let $\varphi = (X; \varphi^-, \varphi^+)$ be a bipolar fuzzy subalgebra of X and let $\alpha_1, \alpha_2 \in [\bot, 0]$ and $\beta_1, \beta_2 \in [0, \top]$. If $(\alpha_1, \beta_1) \geq (\alpha_2, \beta_2)$, i.e., $\alpha_1 \leq \alpha_2$ and $\beta_1 \geq \beta_2$, then the bipolar fuzzy (α_1, β_1)-translation $\varphi^T_{(\alpha_1,\beta_1)} = \left(X; \varphi^-_{(\alpha_1,T)}, \varphi^+_{(\beta_1,T)}\right)$ of $\varphi = (X; \varphi^-, \varphi^+)$ is a bipolar fuzzy S-extension of the bipolar fuzzy (α_2, β_2)-translation $\varphi^T_{(\alpha_2,\beta_2)} = \left(X; \varphi^-_{(\alpha_2,T)}, \varphi^+_{(\beta_2,T)}\right)$ of $\varphi = (X; \varphi^-, \varphi^+)$.*

Proof. Straightforward. □

For every bipolar fuzzy subalgebra $\varphi = (X; \varphi^-, \varphi^+)$ of X and $(\alpha, \beta) \in [\bot, 0] \times [0, \top]$, the bipolar fuzzy (α, β)-translation $\varphi^T_{(\alpha,\beta)} = \left(X; \varphi^-_{(\alpha,T)}, \varphi^+_{(\beta,T)}\right)$ of $\varphi = (X; \varphi^-, \varphi^+)$ is a bipolar fuzzy subalgebra of X. If $\psi = (X; \psi^-, \psi^+)$ is a bipolar fuzzy S-extension of $\varphi^T_{(\alpha,\beta)} = \left(X; \varphi^-_{(\alpha,T)}, \varphi^+_{(\beta,T)}\right)$, then there exists $(k^-, k^+) \in [\bot, 0] \times [0, \top]$ such that $(k^-, k^+) \geq (\alpha, \beta)$, that is, $k^- \leq \alpha$ and $k^+ \geq \beta$ such that $\psi^-(x) \leq \varphi^-_{(k^-,T)}(x)$ and $\psi^+(x) \geq \varphi^+_{(k^+,T)}(x)$ for all $x \in X$. Thus we have the following theorem.

THEOREM 3.13. *Let $\varphi = (X; \varphi^-, \varphi^+)$ be a bipolar fuzzy subalgebra of X and $(\alpha, \beta) \in [\bot, 0] \times [0, \top]$. For every bipolar fuzzy S-extension $\psi = (X; \psi^-, \psi^+)$ of the bipolar fuzzy (α, β)-translation $\varphi^T_{(\alpha,\beta)} = \left(X; \varphi^-_{(\alpha,T)}, \varphi^+_{(\beta,T)}\right)$ of $\varphi = (X; \varphi^-, \varphi^+)$, there exists $(k^-, k^+) \in [\bot, 0] \times [0, \top]$ such that $(k^-, k^+) \geq (\alpha, \beta)$, that is, $k^- \leq \alpha$ and $k^+ \geq \beta$, and $\psi = (X; \psi^-, \psi^+)$ is a bipolar fuzzy S-extension of bipolar fuzzy (k^-, k^+)-translation $\varphi = (X; \varphi^-_{(k^-,T)}, \varphi^+_{(k^+,T)})$ of $\varphi = (X; \varphi^-, \varphi^+)$.*

The following example illustrates Theorem 3.13.

$*$	θ	a	b	c	d
θ	θ	θ	θ	θ	θ
a	a	θ	θ	a	a
b	b	b	θ	b	b
c	c	c	c	θ	c
d	d	d	d	d	θ

TABLE 3. $*$-multiplication table for X

EXAMPLE 3.14. *Consider a BCK-algebra* $X = \{\theta, a, b, c, d\}$ *where the* $*$-*multiplication is defined by Table 3. Let* $\varphi = (X; \varphi^-, \varphi^+)$ *be a bipolar fuzzy subset of* X *defined by*

X	θ	a	b	c	d
φ^-	-0.8	-0.5	-0.3	-0.6	-0.2
φ^+	0.7	0.4	0.2	0.5	0.1

Then φ^+ *is a bipolar fuzzy subalgebra of* X, *and* $(\bot, \top) = (-0.2, 0.3)$. *If we take* $(\alpha, \beta) = (-0.1, 0.2)$, *then the bipolar fuzzy* (α, β)-*translation* $\varphi^T_{(\alpha,\beta)} = \left(X; \varphi^-_{(\alpha,T)}, \varphi^+_{(\beta,T)}\right)$ *of* $\varphi = (X; \varphi^-, \varphi^+)$ *is given by*

X	θ	a	b	c	d
$\varphi^-_{(\alpha,T)}$	-0.9	-0.6	-0.4	-0.7	-0.3
$\varphi^+_{(\beta,T)}$	0.9	0.6	0.4	0.7	0.3

Let $\psi = (X; \psi^-, \psi^+)$ *be a bipolar fuzzy subset of* X *defined by*

X	θ	a	b	c	d
ψ^-	-0.93	-0.66	-0.47	-0.76	-0.42
ψ^+	0.94	0.63	0.55	0.88	0.37

Then $\psi = (X; \psi^-, \psi^+)$ *is clearly a bipolar fuzzy subalgebra of* X *which is bipolar fuzzy extension of* $\varphi^T_{(\alpha,\beta)} = \left(X; \varphi^-_{(\alpha,T)}, \varphi^+_{(\beta,T)}\right)$, *and hence* $\psi = (X; \psi^-, \psi^+)$ *is a bipolar fuzzy* S-*extension of the bipolar fuzzy* (α, β)-*translation* $\varphi^T_{(\alpha,\beta)} = \left(X; \varphi^-_{(\alpha,T)}, \varphi^+_{(\beta,T)}\right)$ *of* $\varphi = (X; \varphi^-, \varphi^+)$. *But* $\psi = (X; \psi^-, \psi^+)$ *is not a bipolar fuzzy* (k^-, k^+)-*translation of* $\varphi = (X; \varphi^-, \varphi^+)$ *for* $(k^-, k^+) \in [\bot, 0] \times [0, \top]$. *Take* $(k^-, k^+) = (-0.12, 0.23)$. *Then* $(k^-, k^+) = (-0.12, 0.23) > (-0.1, 0.2) = (\alpha, \beta)$, *and the bipolar fuzzy* (k^-, k^+)-*translation* $\varphi^T_{(k^-,k^+)} = \left(X; \varphi^-_{(k^-,T)}, \varphi^+_{(k^+,T)}\right)$ *of*

$\varphi = (X; \varphi^-, \varphi^+)$ is given as follows:

X	θ	a	b	c	d
$\varphi^-_{(k^-,T)}$	-0.92	-0.62	-0.42	-0.72	-0.32
$\varphi^+_{(k^+,T)}$	0.93	0.63	0.43	0.73	0.33

Note that $\psi^-(x) \leq \varphi^-_{(k^-,T)}(x)$ and $\psi^+(x) \geq \varphi^+_{(k^+,T)}(x)$ for all $x \in X$, and hence $\psi = (X; \psi^-, \psi^+)$ is a bipolar fuzzy S-extension of the bipolar fuzzy (k^-, k^+)-translation $\varphi^T_{(k^-,k^+)} = \left(X; \varphi^-_{(k^-,T)}, \varphi^+_{(k^+,T)}\right)$ of $\varphi = (X; \varphi^-, \varphi^+)$.

References

[1] Y. Huang, *BCI-algebra,* Science Press, Beijing, 2006.

[2] Y. B. Jun and S. Z. Song, *Subalgebras and closed ideals of BCH-algebras based on bipolar-valued fuzzy sets,* Sci. Math. Jpn. **68** (2008), no. 2, 287–297.

[3] K. J. Lee, *Bipolar fuzzy subalgerbas and bipolar fuzzy ideals of BCK/BCI-algerbas,* Bull. Malays. Math. Sci. Soc. (in press).

[4] K. M. Lee, *Bipolar-valued fuzzy sets and their operations,* Proc. Int. Conf. on Intelligent Technologies, Bangkok, Thailand (2000), 307–312.

[5] K. M. Lee, *Comparison of interval-valued fuzzy sets, intuitionistic fuzzy sets, and bipolar-valued fuzzy sets,* J. Fuzzy Logic Intelligent Systems **14** (2004), no. 2, 125–129.

[6] J. Meng and Y. B. Jun, *BCK-algebras,* Kyungmoon Sa Co. Seoul, 1994.

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