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ON A CLASS OF GENERALIZED LOGARITHMIC FUNCTIONAL EQUATIONS

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Abstract. Reducing the generalized logarithmic functional equations to differential equations in the sense of Schwartz distributions, we find the locally integrable solutions of the equations.

1. Introduction

In this paper we consider the following Pexider generalizations of logarithmic functional equations

$$f(x+y) - g(xy) = h(1/x + 1/y), \quad x, y > 0, \tag{1.1}$$

$$f(x+y) - g(x) - h(y) = k(1/x + 1/y), \quad x, y > 0, \tag{1.2}$$

$$f\left(\frac{x+y}{2}\right) + g\left(\frac{2xy}{x+y}\right) = h(x) + k(y), \quad x, y \in I, \tag{1.3}$$

where $I \subset (0, \infty)$ is an open interval.

As results, reducing the equation (1.1), (1.2) and (1.3) to differential equations, which is one of the most powerful advantages of the Schwartz theory, we find the locally integrable solutions of the equation (1.1), (1.2) and (1.3). We refer the reader to [1, 4, 5] for more results using this method of reducing given functional equations to differential equations.

2. Distributional approach to generalized logarithmic functional equations

Let Ω be an open subset of $\mathbb{R}^n$. We denote by $\mathcal{D}'(\Omega)$ the space of Schwartz distributions on Ω. Recall that a distribution u is a linear functional on $C_c^\infty(\Omega)$ of infinitely differentiable functions on Ω with compact

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supports such that for every compact subset $K \subset \Omega$ there exist constants C and k satisfying

$$|\langle u, \varphi\rangle| \le C \sum_{|\alpha|\le k} \sup |\partial^\alpha \varphi|$$

for all $\varphi \in C_c^\infty(\Omega)$ with supports contained in K. Here we denote by $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$, for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$, where $\mathbb{N}_0$ is the set of non-negative integers and $\partial_j = \frac{\partial}{\partial x_j}$. We briefly introduce some basic operations in $\mathcal{D}'(\Omega)$. Let $u \in \mathcal{D}'(\Omega)$. Then the k-th partial derivative $\partial_k u$ of u is defined by

$$\langle \partial_k u,\, \phi\rangle = -\langle u,\, \partial_k \phi\rangle$$

for $k = 1, \ldots, n$. Let $f \in C^\infty(\Omega)$. Then the multiplication fu is defined by

$$\langle fu,\, \phi\rangle = \langle u,\, f\phi\rangle.$$

We denote by Ω_j open subsets of $\mathbb{R}^{n_j}$ for $j = 1, 2$.

Definition 2.1. Let $u_j \in \mathcal{D}'(\Omega_j)$ and $f : \Omega_1 \to \Omega_2$ a smooth function such that for each $x \in \Omega_1$ the derivative $f'(x)$ is surjective. Then there exists a unique continuous linear map $f^* : \mathcal{D}'(\Omega_2) \to \mathcal{D}'(\Omega_1)$ such that $f^*u = u \circ f$ when u is a continuous function. We call f^*u the pullback of u by f and often denoted by $u \circ f$.

Every locally integrable function $f : (0, \infty) \to \mathbb{C}$ can be viewed as a distribution via the equation

$$\langle f, \varphi\rangle = \int f(x)\varphi(x)\, dx,$$

for all $\varphi \in C_c^\infty((0,\infty))$. Consequently the functional equations (1.1), (1.2) can be viewed in the sense of functional on $C_c^\infty((0,\infty)^2)$ and the functional equation (1.3) can be viewed in the sense of functional on $C_c^\infty(I^2)$. By virtue of the Schwartz theory we can differentiate the locally integrable functions freely in the sense of distributions and reduce the equations to differential equations which is very useful method of finding the solutions of the logarithmic functional equations.

From the above definitions we can check that the usual Leibniz rule and chain rule for differentiations are fulfilled in the space of distributions.

Theorem 2.2. *Every locally integrable solution f, g, h of the functional equation* (1.1) *has the form*

$$f(z) = c_1 + c_2 + a \ln z, \ z > 0 \tag{2.1}$$

$$g(z) = c_1 + a \ln z, \ z > 0 \tag{2.2}$$

$$h(z) = c_2 + a \ln z, \ z > 0 \tag{2.3}$$

in almost everywhere sense, where $a, c_1, c_2 \in \mathbb{C}$.

Proof. We denote by S, P, R, P_1, P_2 the functions $S(x, y) = x + y$, $P(x, y) = xy$, $R(x, y) = 1/x+1/y$, $P_1(x, y) = x$, $P_2(x, y) = y$. Applying $\partial_1 - \partial_2$ in (1.1) we have

$$(x - y)(g' \circ P) = \frac{x^2 - y^2}{x^2y^2}\,(h' \circ R),$$

which implies

$$g' \circ P = \frac{x + y}{x^2y^2}\,(h' \circ R), \tag{2.4}$$

on the space $C_c^\infty(V)$, where $V = \{(x, y) : x > y > 0\}$. Let $J : V \to U := \{s, t) : st^2 > 4,\ t > 0\}$ be a diffeomorphism defined by $J(x, y) = (xy, 1/x + 1/y)$. Then, since $(S \circ J^{-1})(s, t) = s$, $(R \circ J^{-1})(s, t) = t$, taking pullback of (2.4) by J^{-1}, we have

$$(zg'(z)) \circ P_1 = (zh'(z)) \circ P_2 \tag{2.5}$$

as distributions in $\mathcal{D}'(U)$, where $P_1(s, t) = s$, $P_2(s, t) = t$. Applying ∂_1 in (2.5) (or localizing and applying tensor product of test functions) we get

$$zg'(z) = zh'(z) := a. \tag{2.6}$$

Since the solutions $g, h \in \mathcal{D}'((0,\infty))$ of the equation (2.6) are the same as the classical solutions of the equation we get (2.2) and (2.3). Choose $\psi \in C_c^\infty((0,\infty))$ such that $\int \psi(y)dy = 1$. For given $\varphi \in C_c^\infty((0,\infty))$, applying $\phi(x, y) = \varphi(x + y)\psi(y)$ in (1.1) and using (2.2) and (2.3), we get

$$\langle f, \varphi\rangle = \int (c_1 + c_2 + a \ln z)\varphi(z)dz,$$

for all $\varphi \in C_c^\infty((0,\infty))$, which gives (2.1). This completes the proof. □

As a consequence of the above result we have the following.

Corollary 2.3. *Every locally integrable solution f, g, $h : (0,\infty) \to \mathbb{C}$ of the equation*

$$f(x+y) - g(xy) = h(1/x + 1/y), \ \ x, y > 0,$$

has the form

$$f(z) = c_1 + c_2 + a \ln z, \tag{2.7}$$

$$g(z) = c_1 + a \ln z, \tag{2.8}$$

$$h(z) = c_2 + a \ln z, \tag{2.9}$$

where $c_1, c_2, a \in \mathbb{C}$.

Proof. It follows from Theorem 2.2 that the functional equations (2.7), (2.8) and (2.9) hold for all z in a subset $\Omega \subset (0,\infty)$ with $m(\Omega^c) = 0$. For given $z > 0$, let p, $q : (0,\infty) \to \mathbb{R}$ by $p(t) = t + z/t$, $q(t) = 1/t + t/z$. Then since $m(p^{-1}(\Omega^c) \cup q^{-1}(\Omega^c) = 0)$, we have $p^{-1}(\Omega) \cap q^{-1}(\Omega) \neq \emptyset$. Thus we can choose x, $y > 0$ so that $xy = z$ and $x + y$, $1/x + 1/y \in \Omega$. It follows from (1.1), (2.7) and (2.9) that

$$\begin{aligned} g(z) &= f(x+y) - h(1/x + 1/y) \\ &= c_1 + a \ln(xy) = c_1 + a \ln z, \end{aligned} \tag{2.10}$$

which means that the equation (2.8) holds for all $z > 0$. Similarly, let $p : (0, z) \to \mathbb{R}$ by $p(t) = 1/t + 1/(z - t)$. Then we have $p^{-1}(\Omega) \neq \emptyset$. Thus we can choose x, $y > 0$ so that $x + y = z$ and $1/x + 1/y \in \Omega$. Thus it follows from (1.1), (2.7) and (2.8) that the equation(2.9) holds for all $z > 0$, and then, from (1.1) the equation (2.7) also holds for all $z > 0$. This completes the proof. □

Theorem 2.4. *Every locally integrable solution f, g, h, k of the equation* (1.2) *has the form*

$$f(z) = -a \ln z + bz + c_1, \ \ z > 0 \tag{2.11}$$

$$g(z) = -a \ln z + bz - d/z + c_1 + c_3, \ \ z > 0 \tag{2.12}$$

$$h(z) = -a \ln z + bz - d/z - c_2 - c_3, \ \ z > 0 \tag{2.13}$$

$$k(z) = -a \ln z + dz + c_2, \ \ z > 0 \tag{2.14}$$

in almost everywhere sense, where $a, b, d, c_1, c_2, c_3 \in \mathbb{C}$.

Proof. Applying $\partial_1 \partial_2$ in (1.2) we have

$$f'' \circ S = \frac{1}{x^2 y^2} (k'' \circ R). \tag{2.15}$$

Similarly as in the proof of Theorem 2.2, define a diffeomorphism $J : V = \{x, y) : x > y > 0\} \to U := \{s, t) : s > 0,\, st > 4\}$ by $J(x, y) = (x + y, 1/x + 1/y)$, and taking pullback of (2.15) by J^{-1} we have

$$(z^2 f''(z)) \circ P_1 = (z^2 k''(z)) \circ P_2 \tag{2.16}$$

as distributions in $\mathcal{D}(U)$. Applying ∂_1 in (2.16) we get

$$z^2 f''(z) = z^2 k''(z) := a. \tag{2.17}$$

The solution $f, k \in \mathcal{D}'((0, \infty))$ of the equation (2.17) are given by (2.11) and (2.14). It follows from (2.11), (2.14) and (1.2) that

$$\begin{aligned} g \circ P_1 + h \circ P_2 =& f \circ S - k \circ R \\ =& - a \ln(x + y) + b(x + y) + c_1 \\ & + a \ln(1/x + 1/y) - d(1/x + 1/y) - c_2 \\ =& - (a \ln x - bx + d/x - c_1) \\ & - (a \ln y - by + d/y + c_2). \end{aligned} \tag{2.18}$$

Thus it follows that

$$(g + a \ln z - bz + d/z - c_1) \circ P_1 = -(h + a \ln z - bz + d/z + c_2) \circ P_2,$$

and hence

$$g + a \ln z - bz + d/z - c_1 = -(h + a \ln z - bz + d/z + c_2) := c_3,$$

which gives (2.12) and (2.13). This completes the proof.

□

As a consequence of Theorem 2.3 we have the following.

Corollary 2.5. *Every locally integrable solution* $f,\ g,\ h,\ k : (0, \infty) \to \mathbb{C}$ *of the functional equation*

$$f(x + y) - g(x) - h(y) = k(1/x + 1/y)$$

has the form

$$f(z) = -a \ln z + bz + c_1, \tag{2.19}$$

$$g(z) = -a \ln z + bz - d/z + c_1 + c_3, \tag{2.20}$$

$$h(z) = -a \ln z + bz - d/z - c_2 - c_3, \tag{2.21}$$

$$k(z) = -a \ln z + dz + c_2, \tag{2.22}$$

where $a, b, d, c_1, c_2, c_3 \in \mathbb{R}$.

Proof. From the above result the equation (2.19) $\sim$ (2.22) hold for all z in a subset $\Omega \subset (0,\infty)$ with $m(\Omega^c) = 0$. For given $x > 0$, let $p, q : (0,\infty) \to \mathbb{R}$ by $p(t) = x + t$, $q(t) = 1/x + 1/t$. Then since $\Omega \cap p^{-1}(\Omega) \cap q^{-1}(\Omega) \neq \emptyset$, we can choose $y > 0$ so that y, $x+y$, $1/x+1/y \in \Omega$. It follows from (1.2), (2.19), (2.21) and (2.22) that

$$\begin{aligned} g(x) =& f(x+y) - h(y) - k(1/x+1/y) \\ =& - a\ln(x+y) + b(x+y) + c_1 \\ & + a\ln y - by + d/y + c_2 + c_3 \\ & + a\ln(1/x+1/y) - d(1/x+1/y) - c_2 \\ =& - a\ln x + bx - d/x + c_1 + c_3, \end{aligned}$$

which means that the equation (2.20) holds for all $z > 0$. Exchanging x and y in (1.2) and following the same approach as above we get the equality (2.21). Similarly, for given $z > 0$, we can choose $x, y \in \Omega$ so that $x+y = z$, $1/x+1/y \in \Omega$ and the equation (2.19) follows from (1.2), (2.20), (2.21) and (2.22). Finally, the equation (2.22) follows from (1.2) and the equalities (2.19), (2.20) and (2.21). □

We denote by M, H the functions $M(x,y) = (x+y)/2$, $H(x,y) = 2xy/(x+y)$.

THEOREM 2.6. *Every locally integrable solution f, g, h, k of the functional equation* (1.3) *has the form*

$$f(z) = a\ln z + b\ln z + c_1,\ z \in I \tag{2.23}$$

$$g(z) = b\ln z + c_1 + c_2 - c_3,\ z \in I \tag{2.24}$$

$$h(z) = \frac{a}{2}\ln z + b\ln z - d/z + c_2,\ z \in I \tag{2.25}$$

$$k(z) = \frac{a}{2}\ln z + b\ln z - d/z + c_3,\ z \in I \tag{2.26}$$

in almost everywhere sense, where $a, b, d, c_1, c_2, c_3 \in \mathbb{C}$.

Proof. Applying $x^2\partial_1 - y^2\partial_2$ in (1.3) we have

$$\frac{1}{2}(x^2 - y^2)(f' \circ M) = x^2(h' \circ P_1) - y^2(k' \circ P_2). \tag{2.27}$$

If we denote by f^*, h^*, k^*,

$$f^*(z) = zf'(z),\ h^*(z) = z^2h'(z),\ k^*(z) = z^2k'(z), \tag{2.28}$$

the equation (2.27) is written by

$$(x-y)(f^* \circ M) = h^* \circ P_1 - k^* \circ P_2. \tag{2.29}$$

Applying $\partial_1 - \partial_2$ in (2.29) we have

$$2(f^* \circ M) = (h^*)' \circ P_1 + (k^*)' \circ P_2. \tag{2.30}$$

As a local version of the Jensen-Pexider type functional equation[2, Theorem 3.4], it is easy to see that the solution f^*, $(h^*)'$, $(k^*)'$ have the form

$$f^*(z) = az + (b+c)/2,\ (h^*)'(z) = az + b,\ (k^*)'(z) = az + c, \tag{2.31}$$

and it follows that

(2.32)

$$f^*(z) = az + (b+c)/2,\ h^*(z) = \frac{a}{2}z^2 + bz + d_1,\ k^*(z) = \frac{a}{2}z^2 + cz + d_2.$$

It follows from (2.29) and (2.32) that

$$\left(\frac{b+c}{2}\right)(x-y) = bx - cy + d_1 - d_2$$

as distributions in $\mathcal{D}'(I^2)$, which implies $b = c$, $d_1 = d_2 := d$. Thus we have

$$f^*(z) = az + b,\ h^*(z) = k^*(z) = \frac{a}{2}z^2 + bz + d. \tag{2.33}$$

In view of (2.28) and (2.33) we have (2.23), (2.25) and (2.26). Putting the equations (2.23), (2.25) and (2.26) in (1.3) we get (2.24). This completes the proof.

□

Similarly as in the proof of Corollary 2.3 and Corollary 2.5 we have the followings.

Corollary 2.7. *Every locally integrable solution f, g, h, $k : I \to \mathbb{C}$ of the functional equation*

$$f\left(\frac{x+y}{2}\right) + g\left(\frac{2xy}{x+y}\right) = h(x) + k(y),\ x, y \in I$$

has the form

$$f(z) = b\ln z + az + c_1, \tag{2.34}$$

$$g(z) = b\ln z + 2d/z + c_2 + c_3 - c_1, \tag{2.35}$$

$$h(z) = b\ln z + \frac{a}{2}z + d/z + c_2, \tag{2.36}$$

$$k(z) = \frac{a}{2}z + b\ln z + d/z + c_3, \tag{2.37}$$

where $a, b, d, c_1, c_2, c_3 \in \mathbb{C}$.

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