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UNIQUENESS AND MULTIPLICITY OF SOLUTIONS FOR THE NONLINEAR ELLIPTIC SYSTEM

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Abstract. We investigate the uniqueness and multiplicity of solutions for the nonlinear elliptic system with Dirichlet boundary condition

$$\begin{cases} -\Delta u + g_1(u,v) = f_1(x) & \text{in } \Omega, \\ -\Delta v + g_2(u,v) = f_2(x) & \text{in } \Omega, \end{cases}$$

where Ω is a bounded set in R^n with smooth boundary $\partial\Omega$. Here g_1, g_2 are nonlinear functions of u, v and g_1, g_2 are nonlinear functions of u, v and f_1, f_2 are source terms.

1. Introduction

In this paper we investigate the uniqueness and multiplicity of solutions for the nonlinear elliptic system with Dirichlet boundary condition

$$\begin{cases} -\Delta u + g_1(u,v) = f_1(x) & \text{in } \Omega, \\ -\Delta v + g_2(u,v) = f_2(x) & \text{in } \Omega, \\ u = 0, \qquad v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded set in R^n with smooth boundary $\partial\Omega$. Here g_1, g_2 are nonlinear functions of u, v and f_1, f_2 are source terms.

System (1.1) can be rewritten by

$$\begin{cases} -\Delta U + G(U) = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} & \text{in } \Omega, \\ U = \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $U = \binom{u}{v}$, $G(U) = \binom{g_1}{g_2}$, $-\Delta U = \binom{-\Delta u}{-\Delta v}$.

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System (1.1) of the nonlinear biharmonic equations with Dirichlet boundary condition is considered as a model of the cross of the two nonlinear oscillations in differential equation.

For the case of the single biharmonic equation Tarantello([9]), Lazer and McKenna([7]), Choi and Jung ([4]) etc., investigate the multiplicity of the solutions via the degree theory or the critical point theory or the variational reduction method. In this paper we improve the multiplicity results of the single biharmonic equation to the case of the system of the nonlinear elliptic system.

Let Ω be a bounded set in R^n with smooth boundary $\partial\Omega$. Let λ_k, $k = 1, 2, \ldots$, denote the eigenvalues and ϕ_k, $k = 1, 2, \ldots$, the corresponding eigenfunctions, suitably normalized with respect to $L^2(\Omega)$ inner product, of the eigenvalue problem

$$\Delta u + \lambda u = 0 \qquad \text{in } \Omega,$$

$$u = 0 \qquad \text{on } \partial\Omega,$$

where each eigenvalue λ_k is repeated as often as its multiplicity. We recall that $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \ldots \to +\infty$, and that $\phi_1(x) > 0$ for $x \in \Omega$. The set of functions $\{\phi_k\}$ is an orthonormal base for $L^2(\Omega)$. Let us denote an element u, in $L^2(\Omega)$, as

$$u = \sum h_k \phi_k, \qquad \sum h_k^2 < \infty.$$

We define a subspace $\mathcal{D}$ of $L^2(\Omega)$ as follows

$$\mathcal{D} = \{u \in L^2(\Omega) | \sum \lambda_k h_k^2 < \infty\}.$$

Then this is a complete normed space with a norm

$$\|u\| = [\sum \lambda_k h_k^2]^{\frac{1}{2}}.$$

Let us set $E = \mathcal{D} \times \mathcal{D}$. We endow the Hilbert E with the norm

$$\|(u, v)\|_E^2 = \|u\|^2 + \|v\|^2 \qquad \forall (u, v) \in E.$$

We are looking for the weak solutions of (1.1) in E, that is, (u, v) satisfying the equation

$$\int_\Omega (-\Delta u + g_1(u, v))z + \int_\Omega (-\Delta v + g_2(u, v))w - \int_\Omega f_1 z - \int_\Omega f_2 w = 0$$

for all $(z, w) \in E$.

In section 2 we investigate the uniqueness of solutions for the nonlinear elliptic system with Dirichlet boundary condition

$$\begin{cases} -\Delta u + av^+ = \alpha\phi_1 + f & \text{in } \Omega, \\ -\Delta v + bu^+ = \beta\phi_1 + g & \text{in } \Omega, \\ u = 0, \quad v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where $u^+ = \max\{u, 0\}$, $a, b \in R$, $\alpha, \beta \in R$,. Here $\phi_1$is the positive eigenfunction of the eigenvalue problem $\Delta u + \lambda u = 0 \quad \text{in } \Omega$, $u = 0 \quad \text{on } \partial\Omega$ and λ_1 is the first eigenvalue corresponding to ϕ_1. In section 3 we investigate the multiplicity of solutions for the nonlinear elliptic system with Dirichlet boundary condition

$$\begin{cases} -\Delta u + av^- = \alpha\phi_1 + \epsilon_1 f & \text{in } \Omega, \\ -\Delta v + bu^- = \beta\phi_1 + \epsilon_2 g & \text{in } \Omega, \\ u = 0, \quad v = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.4}$$

2. Uniqueness result for the elliptic system

In this section we investigate the uniqueness of solutions for the nonlinear elliptic system with Dirichlet boundary condition

$$\begin{cases} -\Delta u + av^+ = \alpha\phi_1 + f & \text{in } \Omega, \\ -\Delta v + bu^+ = \beta\phi_1 + g & \text{in } \Omega, \\ u = 0, \quad v = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where $u^+ = \max\{u, 0\}$, $a, b \in R$, $\alpha, \beta \in R$,. Here $\phi_1$is the positive eigenfunction of the eigenvalue problem $\Delta u + \lambda u = 0 \quad \text{in } \Omega$, $u = 0 \quad \text{on } \partial\Omega$ and λ_1 is the first eigenvalue corresponding to ϕ_1.

The subspace $\mathcal{D}$ of $L^2(\Omega)$,

$$\mathcal{D} = \{u \in L^2(\Omega) | \sum \lambda_k h_k^2 < \infty\},$$

a complete normed space with a norm

$$\|u\| = [\sum \lambda_k h_k^2]^{\frac{1}{2}}.$$

Let us set $E = \mathcal{D} \times \mathcal{D}$. We endow the Hilbert space E with the norm

$$\|(u, v)\|_E^2 = \|u\|^2 + \|v\|^2.$$

We are looking for the weak solutions of (2.1) in $\mathcal{D} \times \mathcal{D}$, that is, (u, v) such that $u \in \mathcal{D}$, $v \in \mathcal{D}$, $Lu + av^+ = \alpha\phi_1 + f$, $Lv + au^+ = \beta\phi_1 + g$.

Lemma 2.1. *Suppose that c is not an eigenvalue of $L : \mathcal{D} \to H_0$, $Lu = u - \Delta u$, and let $f \in \mathcal{D}$. Then we have $(L-c)^{-1} f \in \mathcal{D}$.*

Lemma 2.1 was proved in [3].

Lemma 2.2. *The system*

$$\begin{cases} -\Delta u + av = \alpha\phi_1 & \text{in } \Omega, \\ -\Delta v + bu = \beta\phi_1 & \text{in } \Omega, \\ u = 0, \quad v = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.2}$$

has a unique solution $(u^*, v^*) \in E = \mathcal{D} \times \mathcal{D}$, which is of the form

$$u^* = \frac{\alpha\lambda_1 - b\beta}{\lambda_1^2 - ab}\phi_1, \qquad v^* = \frac{\beta\lambda_1 - a\alpha}{\lambda_1^2 - ab}\phi_1.$$

Proof. We note that (u^*, v^*) is a solution of the system (2.2) and the uniqueness is trivial. □

We need to find a spectral analysis for the linear operator $-\Delta U$. The following lemma need a simple 'Fourier Series' argument.

Lemma 2.3. *Let a, $b \in R$ and let $\mathcal{L}_{ab} : \mathcal{D} \times \mathcal{D} \to L^2(\Omega) \times L^2(\Omega)$ be defined by $\mathcal{L}_{ab}(u, v) = (Lu + av, Lv + bu)$. For $\mu \in R$ we have*
(a) if $(\lambda_j - \mu)^2 \neq ab$ for every j, then

$$(\mathcal{L}_{ab} - \mu I)^{-1} : L^2(\Omega) \times L^2(\Omega) \to L^2(\Omega) \times L^2(\Omega)$$

is well defined and continuous;
(b) if $(\lambda_j - \mu)^2 = ab$ for some j, then

$$\mathrm{Ker}(\mathcal{L}_{ab} - \mu I) = \mathrm{span}\{\phi_j : \ (\lambda_j - \mu)^2 = ab\};$$

moreover if $X_\mu = \overline{\mathrm{span}\{\phi_{mn} : \ (\lambda_{mn} - \mu)^2 \neq ab\}}$, then

$$(\mathcal{L}_{ab} - \mu I)^{-1} : X_\mu \times X_\mu \to X_\mu \times X_\mu$$

is well defined and continuous.
Notice that if $ab < 0$, the second alternative can never occur.

For the proof of the lemma we refer [3].
We assume that

$$\lambda_j^2 - ab \neq 0, \qquad \text{for all } j \text{ with } j \geq 0, \tag{2.3}$$

$$a < \lambda_1, \qquad b < \lambda_1, \tag{2.4}$$

$$ab > 0, \qquad \sqrt{ab} < \lambda_1. \tag{2.5}$$

Using Lemma 2.3 with the case $\mathcal{L}(u,v) = (Lu, Lv)$ we can easily derive the following lemma.

Lemma 2.4. *Assume that the conditions (2.3), (2.4) and (2.5) hold. Then the system*

$$\begin{cases} -\Delta u + av = 0 & \text{in } \Omega, \\ -\Delta v + bu = 0 & \text{in } \Omega, \\ u = 0, \qquad v = 0 & \text{on } \partial\Omega, \end{cases}$$

has only the trivial solution $U = \binom{0}{0}$.

Lemma 2.5. *Assume that* $f,\ g \in \mathcal{D}$ *with* $\int_\Omega f\phi_1 = \int_\Omega g\phi_1 = 0$. *Then the system*

$$\begin{cases} -\Delta u + av = f & \text{in } \Omega, \\ -\Delta v + bu = g & \text{in } \Omega, \\ u = 0, \qquad v = 0 & \text{on } \partial\Omega, \end{cases}$$

has a unique solution $(u_0, v_0) \in E = \mathcal{D} \times \mathcal{D}$.

Theorem 2.6. *(Existence of a negative solution) Assume that the conditions (2.3), (2.4) and (2.5) hold. Assume that* $f,\ g \in L^2(\Omega)$ *with* $\int_\Omega f\phi_1 = \int_\Omega g\phi_1 = 0$. *Then there exists* (α_0, β_0) *with* $\alpha_0 < 0$ *and* $\beta_0 < 0$ *such that the system (2.1) has a negative solution* $(\check{u}, \check{v})$ *with* $\check{u} < 0$ *and* $\check{v} < 0$ *for each* α *and* β *with* $\alpha < \alpha_0$ *and* $\beta < \beta_0$,

Proof. By Lemma 2.2 and Lemma 2.5, $(u^* + u_0, v^* + v_0)$ is a solution of the system

$$\begin{cases} -\Delta u + av = \alpha\phi_1 + f & \text{in } \Omega, \\ -\Delta v + bu = \beta\phi_1 + g & \text{in } \Omega, \\ u = 0, \qquad v = 0 & \text{on } \partial\Omega. \end{cases}$$

By Lemma 2.6, $u_0 \in \mathcal{D}$ and $v_0 \in \mathcal{D}$. Since the elements of $\mathcal{D}$ lies in C^1, the elements $u_0,\ v_0 \in C^1$. Thus we can find (α_0, β_0) with $\alpha_0 < 0$ and $\beta_0 < 0$ such that $u^* + u_0 < 0$ and $v^* + v_0 < 0$ for each $\alpha < \alpha_0$ and $\beta < \beta_0$. Thus we prove the theorem . □

Theorem 2.7. *(Existence of a positive solution) Assume that the conditions (2.3), (2.4) and (2.5) hold. Assume that* $f,\ g \in L^2(\Omega)$ *with* $\int_\Omega f\phi_1 = \int_\Omega g\phi_1 = 0$. *Then there exists* (α_1, β_1) *with* $\alpha_1 > 0$ *and* $\beta_1 > 0$ *such that system (2.1) has a positive solution* $(\hat{u}, \hat{v})$ *with* $\hat{u} > 0$ *and* $\hat{v} > 0$ *for each* α *and* β *with* $\alpha > \alpha_1$ *and* $\beta > \beta_1$

Proof. By Lemma 2.2 and Lemma 2.5, $(u^* + u_0, v^* + v_0)$ is a solution of the system

$$\begin{cases} -\Delta u + av = \alpha\phi_1 + f & \text{in } \Omega, \\ -\Delta v + bu = \beta\phi_1 + g & \text{in } \Omega, \\ u = 0, \quad v = 0 & \text{on } \partial\Omega. \end{cases}$$

By Lemma 2.5, $u_0 \in \mathcal{D}$ and $v_0 \in \mathcal{D}$. Since the elements of $\mathcal{D}$ lies in C^1, the elements u_0, $v_0 \in C^1$. Thus we can find (α_1, β_1) with $\alpha_1 < 0$ and $\beta_1 < 0$ such that $u^* + u_0 < 0$ and $v^* + v_0 < 0$ for each $\alpha < \alpha_1$ and $\beta < \beta_1$. Thus we prove the theorem . □

Theorem 2.8. *(Uniqueness Theorem) Assume that the conditions (1.2), (1.3) and (1.4) hold and f, $g \in L^2(\Omega)$ with $\int_\Omega f\phi_1 = \int_\Omega g\phi_1 = 0$. Then, (i) system (2.1) has a unique solution in $\mathcal{D} \times \mathcal{D}$. In particular, (ii) there exists (α_0, β_0) with $\alpha_0 < 0$ and $\beta_0 < 0$ such that system (2.1) has a unique solution, which is a negative solution $(\check{u}, \check{v})$ with $\check{u} < 0$ and $\check{v} < 0$ in Theorem 2.6 for each α and β with $\alpha < \alpha_0$ and $\beta < \beta_0$, (iii) there exists (α_1, β_1) with $\alpha_1 > 0$ and $\beta_1 > 0$ such that system (2.1) has a unique solution, which is a positive solution $(\hat{u}, \hat{v})$ with $\hat{u} > 0$ and $\hat{v} > 0$ in Theorem 2.7 for each α and β with $\alpha > \alpha_1$ and $\beta > \beta_1$.*

Proof. Assume that conditions (2.3), (2.4) and (2.5) hold. First we will prove (i). To prove it we use the contraction mapping principle. By assumption (2.5), $-\lambda_1 < -\sqrt{ab} < 0 < \sqrt{ab} < \lambda_1$. Let us set $\delta = \lambda_1$. Then system (2.1) is equivalent to

$$U = (\mathcal{L} + \delta I)^{-1}[(\delta I - A)U^+ - \delta I U^- + \begin{pmatrix} \alpha\phi_1 + f \\ \beta\phi_1 + g \end{pmatrix}], \tag{2.6}$$

where $A = \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}$, $U^+ = \begin{pmatrix} u^+ \\ v^+ \end{pmatrix}$, $U^- = \begin{pmatrix} u^- \\ v^- \end{pmatrix}$ and $(\mathcal{L} + \delta)^{-1}$ is a compact, self-adjoint, linear map from $L^2(\Omega) \times L^2(\Omega)$ into $L^2(\Omega) \times L^2(\Omega)$ with norm $\frac{1}{2\lambda_1}$. We note that

$$\begin{aligned} \|(\delta I - A)(U_2^+ - U_1^+) - \delta I(U_2^- - U_1^-)\| &\le \max\{det(\delta I - A), det(\delta I)\}\|U_2 - U_1\| \\ &< \|U_2 - U_1\|. \end{aligned}$$

It follows that the right hand side of (2.6) defines a Lipschitz mapping of $L^2(\Omega) \times L^2(\Omega)$ into $L^2(\Omega) \times L^2(\Omega)$ with Lipschitz constant $\gamma < 1$. Therefore, by the contraction mapping principle, there exists a unique solution $U = \begin{pmatrix} u \\ v \end{pmatrix} \in L^2(\Omega) \times L^2(\Omega)$ of (2.6). By Lemma 2.1, $U = \begin{pmatrix} u \\ v \end{pmatrix} \in \mathcal{D} \times \mathcal{D}$. Thus (i) is proved and (ii) and (iii) come from Theorem 2.6 and Theorem 2.7. □

3. Multiple solutions for the elliptic system

In this section we investigate the multiplicity of solutions for the nonlinear elliptic system with Dirichlet boundary condition

$$\begin{cases} -\Delta u + av^- = \alpha\phi_1 + \epsilon_1 f & \text{in } \Omega, \\ -\Delta v + bu^- = \beta\phi_1 + \epsilon_2 g & \text{in } \Omega, \\ u = 0, \quad v = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.1}$$

Here we assume that $\alpha > 0$, $\beta > 0$.

LEMMA 3.1. *Assume that* $\alpha > 0$, $\beta > 0$. *Assume that* $(\lambda_1^2 - ab)(\lambda_1\beta - \alpha b) < 0$, $(\lambda_1^2 - ab)(\lambda_1\alpha - \beta a) < 0$. *Then the system*

$$\begin{cases} -\Delta u + av^- = \alpha\phi_1 & \text{in } \Omega, \\ -\Delta v + bu^- = \beta\phi_1 & \text{in } \Omega, \\ u = 0, \quad v = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.2}$$

has at least two solutions, one of which is positive, and one of which is positive.

Proof. Assume that $\alpha > 0$, $\beta > 0$. Then system (3.2) has a positive solution $U_1 = \binom{u_1}{v_1}$:

$$u_1 = \frac{\alpha}{\lambda_1}\phi_1, \quad v_1 = \frac{\beta}{\lambda_1}\phi_1.$$

Since $(\lambda_1^2 - ab)(\lambda_1\beta - \alpha b) < 0$, $(\lambda_1^2 - ab)(\lambda_1\alpha - \beta a) < 0$, system (3.2) has a negative solution $U_2 = \binom{u_2}{v_2}$:

$$u_2 = \frac{\lambda_1\beta - \alpha b}{\lambda_1^2 - ab}\phi_1, \quad v_2 = \frac{\lambda_1\alpha - \beta a}{\lambda_1^2 - ab}\phi_1.$$

Hence (3.2) has at least two solutions, one of which is positive, and one of which is positive. □

THEOREM 3.2. *(Existence of two solutions) Assume that* $\alpha > 0$, $\beta > 0$. *Assume that* $(\lambda_1^2 - ab)(\lambda_1\beta - \alpha b) < 0$, $(\lambda_1^2 - ab)(\lambda_1\alpha - \beta a) < 0$. *Let* $\|f\| = \|g\| = 1$. *Then there exists* $(\epsilon_1^*, \epsilon_2^*)$ *such that if* $\epsilon_1 < \epsilon_1^*, \epsilon_2 < \epsilon_2^*$ *then system (3.1) has at least two solutions, one of which is positive, and one of which is positive.*

The proof of Theorem 3.2 is similar to that of Theorem 2.8.

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