

## ON $h$ -STABILITY OF LINEAR DYNAMIC EQUATIONS ON TIME SCALES VIA $u_\infty$ -SIMILARITY

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ABSTRACT. In this paper we investigate  $h$ -stability for linear dynamic equations on time scales under  $u_\infty$ -similarity.

### 1. Introduction

The concept of  $t_\infty$ -similarity by Conti [14] is an effective tool to study the stability for differential systems. Conti [14] showed that  $t_\infty$ -similarity is an equivalence relation preserving strict, uniform and exponential stability of linear homogeneous differential systems. Hewer [17] studied the variational stability of nonlinear differential systems using the notion of  $t_\infty$ -similarity. Choi et al. [4] investigated  $h$ -stability for the nonlinear differential systems using the notions of  $t_\infty$ -similarity and Liapunov functions.

Trench [20] introduced summable similarity as a discrete analog of Conti's definition of  $t_\infty$ -similarity. Choi and Koo [5] studied the variational stability for nonlinear difference systems by using  $n_\infty$ -similarity. Furthermore, Choi and Koo [13] introduced  $u_\infty$ -similarity in order to unify  $t_\infty$ -similarity and  $n_\infty$ -similarity and then studied the strong stability for linear dynamic equations on time scales by using the concept of  $u_\infty$ -similarity and Gronwall's inequality.

In this paper we investigate the  $h$ -stability for linear dynamic equations on time scales by using the concept of  $u_\infty$ -similarity and Gronwall's inequality.

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## 2. Preliminaries

For an introductory text and an article on the calculus on time scales we refer to Bohner and Peterson [3] and Hilger [16], respectively. The symbol  $|\cdot|$  will be used to denote any convenient vector norm on  $\mathbb{R}^n$  or  $\mathbb{R}^{n^2}$ .

A *time scale*  $\mathbb{T}$  is a nonempty closed subset of  $\mathbb{R}$ , and the *forward jump operator*  $\sigma : \mathbb{T} \rightarrow \mathbb{T}$  is defined by  $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$ , while the *graininess*  $\mu : \mathbb{T} \rightarrow \mathbb{R}_+$  is given by  $\mu(t) = \sigma(t) - t$ . Assume throughout that  $\mathbb{T}$  is unbounded above and the graininess  $\mu$  is bounded.

A function  $f : \mathbb{T} \rightarrow \mathbb{R}$  is called *differentiable* (in a point  $t \in \mathbb{T}$ ), if there exists a unique derivative  $f^\Delta(t) \in \mathbb{R}$ , such that for any  $\varepsilon > 0$  the estimate

$$|f^\sigma(t) - f(s) - f^\Delta(t)(\sigma(t) - s)| \leq \varepsilon|\sigma(t) - s| \text{ for all } s \in U,$$

where  $f^\sigma = f \circ \sigma$ , holds in a  $\mathbb{T}$ -neighborhood  $U$  of  $t$ .

EXAMPLE 2.1. Let  $\delta$  be a positive constant and  $\delta\mathbb{Z} = \{0, \pm\delta, \pm2\delta, \dots\}$ . The derivative  $f^\Delta(t) \in \mathbb{R}$  of a function  $f : \mathbb{T} \rightarrow \mathbb{R}$  reads as

$$f^\Delta(t) = f'(t) \text{ if } \mathbb{T} = \mathbb{R}, \quad f^\Delta(t) = \frac{f(t + \delta) - f(t)}{\delta} \text{ if } \mathbb{T} = \delta\mathbb{Z}.$$

A function  $f : \mathbb{T} \rightarrow \mathbb{R}$  is called *rd-continuous* (denoted by  $f \in C_{\text{rd}}(\mathbb{T}, \mathbb{R})$ ) if

- (i)  $f$  is continuous at every right-dense point  $t \in \mathbb{T}$ ,
- (ii)  $\lim_{s \rightarrow t^-} f(s)$  exists and is finite at every left-dense point  $t \in \mathbb{T}$ .

A function  $g : \mathbb{T} \rightarrow \mathbb{R}$  is called an *antiderivative* of  $f$  on  $\mathbb{T}$  if it is differentiable on  $\mathbb{T}$  and satisfies  $g^\Delta(t) = f(t)$  for  $t \in \mathbb{T}$ . In this case, we define the *Cauchy integral* of  $f$  as  $\int_a^t f(s)\Delta s = g(t) - g(a)$  for  $t, a \in \mathbb{T}$ .

Let  $M_n(\mathbb{R})$  be the set of all  $n \times n$  matrices over  $\mathbb{R}$  and  $\mathfrak{M}_n(\mathbb{R})$  the set of all  $n \times n$  invertible matrices over  $\mathbb{R}$ .

An operator  $A : \mathbb{T} \rightarrow M_n(\mathbb{R})$  is called *regressive* if for each  $t \in \mathbb{T}$  the  $n \times n$  matrix  $I + \mu(t)A(t)$  is invertible. The set of all rd-continuous and regressive operators from  $\mathbb{T}$  to  $M_n(\mathbb{R})$  is denoted by  $C_{\text{rd}}\mathcal{R}(\mathbb{T}, M_n(\mathbb{R}))$ .

We consider linear dynamic systems

$$(2.1) \quad x^\Delta(t) = A(t)x(t), \quad A \in C_{\text{rd}}\mathcal{R}(\mathbb{T}, M_n(\mathbb{R})),$$

and

$$(2.2) \quad y^\Delta(t) = B(t)y(t), \quad B \in C_{\text{rd}}\mathcal{R}(\mathbb{T}, M_n(\mathbb{R})).$$

We recall the notion of  $u_\infty$ -similarity on time scales in order to unify continuous similarity and discrete similarity in [13].

DEFINITION 2.2. Let  $A, B \in C_{\text{rd}}\mathcal{R}(\mathbb{T}, M_n(\mathbb{R}))$  and  $t_0 \in \mathbb{T}$ . An operator  $A$  is  $u_\infty$ -similar to an operator  $B$  if there exists an absolutely integrable operator  $F \in C_{\text{rd}}(\mathbb{T}, M_n(\mathbb{R}))$ , i.e.,  $\int_{t_0}^\infty |F(t)|\Delta t < \infty$ , such that

$$(2.3) \quad S^\Delta(t) + S^\sigma(t)B(t) - A(t)S(t) = F(t), \quad t \in \mathbb{T},$$

for both bounded operators  $S$  and  $S^{-1} \in C_{\text{rd}}^1(\mathbb{T}, \mathfrak{M}_n(\mathbb{R}))$ .

We say that if  $A$  is  $u_\infty$ -similar to  $B$ , then system (2.1) is  $u_\infty$ -similar to system (2.2). We note that  $u_\infty$ -similarity is an equivalence relation.

REMARK 2.3. If  $\mathbb{T} = \mathbb{R}$ , then  $u_\infty$ -similarity becomes  $t_\infty$ -similarity [4, 17] and if  $\mathbb{T} = \mathbb{Z}$ , then  $u_\infty$ -similarity becomes  $n_\infty$ -similarity [5, 6, 8, 20]. Also if  $A$  and  $B$  are  $u_\infty$ -similar with  $F = 0$  in (2.3), then they are kinematically similar [2].

LEMMA 2.4. [13] Assume that  $A$  and  $B$  are  $u_\infty$ -similar. Then

$$\begin{aligned} \Phi_B(t, t_0) &= S^{-1}(t)[\Phi_A(t, t_0)S(t_0) \\ &+ \int_{t_0}^t \Phi_A(t, \sigma(s))F(s)\Phi_B(s, t_0)\Delta s], \quad t, t_0 \in \mathbb{T}, \end{aligned}$$

where  $\Phi_A(t, t_0)$  and  $\Phi_B(t, t_0)$  are the matrix exponential functions of (2.1) and (2.2), respectively.

We can obtain the following results as special cases of Lemma 2.4.

REMARK 2.5. Assume that  $A$  and  $B$  are  $u_\infty$ -similar on  $\mathbb{T}$ .

(i) If  $\mathbb{T} = \mathbb{R}$ , then  $u_\infty$ -similarity becomes  $t_\infty$ -similarity and we obtain

$$\Phi_B(t, t_0) = S^{-1}(t)[\Phi_A(t, t_0)S(t_0) + \int_{t_0}^t \Phi_A(t, s)F(s)\Phi_B(s, t_0)ds].$$

(ii) If  $\mathbb{T} = \mathbb{Z}$ , then  $u_\infty$ -similarity becomes  $n_\infty$ -similarity and we obtain

$$\Phi_B(t, t_0) = S^{-1}(t)[\Phi_A(t, t_0)S(t_0) + \sum_{s=t_0}^{t-1} \Phi_A(t, s+1)F(s)\Phi_B(s, t_0)].$$

### 3. Main results

We consider the dynamic system

$$(3.1) \quad x^\Delta = F(t, x), \quad x(t_0) = x_0,$$

where  $F \in C_{\text{rd}}(\mathbb{T} \times \mathbb{R}^n, \mathbb{R}^n)$  with  $F(t, 0) = 0$  and  $x^\Delta$  is the delta derivative of  $x : \mathbb{T} \rightarrow \mathbb{R}^n$  with respect to  $t \in \mathbb{T}$ . We assume that the solutions  $x$  of (3.1) exist and are unique for  $t \geq t_0$ .

Pinto introduced the notion of  $h$ -stability which is an extension of the notions of exponential stability and uniform Lipschitz stability of differential equations in [19] and difference equations in [18].

We recall the notion of  $h$ -stability of dynamic equations on time scales in [7].

DEFINITION 3.1. System (3.1) is said to be

(i) an  $h$ -system if there exist a positive function  $h : \mathbb{T} \rightarrow \mathbb{R}$  and a constant  $c \geq 1$  such that

$$|x(t, t_0, x_0)| \leq c|x_0|h(t)h(t_0)^{-1}, \quad t \geq t_0$$

for  $|x_0|$  small enough (here  $h(t)^{-1} = \frac{1}{h(t)}$ ).

(ii)  $h$ -stable if system (3.1) is an  $h$ -system and  $h$  is bounded.

Choi et al. [7, 9, 10] and DaCunha [15] gave the characterizations of the various types of stability for solutions of dynamic systems on time scales.

LEMMA 3.2. [15] Let  $\Phi_A(t, t_0)$  be a matrix exponential function of (2.1). Then system (2.1) is

(i) uniformly stable if and only if there exists a positive constant  $\gamma$  such that

$$|\Phi_A(t, t_0)| \leq \gamma, \quad t \geq t_0 \in \mathbb{T}.$$

(ii) uniformly exponential stable if and only if there exist positive constants  $\gamma, \lambda$  with  $-\lambda \in \mathcal{R}^+$  such that

$$|\Phi_A(t, t_0)| \leq \gamma e_{-\lambda}(t, t_0), \quad t \geq t_0 \in \mathbb{T}.$$

(iii) uniformly exponential stable if and only if it is uniformly asymptotically stable.

LEMMA 3.3. [7, Lemma 2.3] System (2.1) is an  $h$ -system if and only if there exist a positive function  $h$  defined on  $\mathbb{T}$  and a constant  $c \geq 1$  such that

$$|\Phi_A(t, t_0)| \leq ch(t)h(t_0)^{-1}, \quad t \geq t_0 \in \mathbb{T}.$$

THEOREM 3.4. Assume that systems (2.1) and (2.2) are  $u_\infty$ -similar with  $\int_{t_0}^\infty \frac{h(s)}{h(\sigma(s))} |F(s)| \Delta s < \infty$  for each  $t_0 \in \mathbb{T}$ . Then (2.1) is an  $h$ -system if and only if (2.2) is also an  $h$ -system.

*Proof.* Suppose (2.1) is an  $h$ -system. Then, there exist a positive function  $h$  defined on  $\mathbb{T}$  and a constant  $c \geq 1$  such that

$$|\Phi_A(t, t_0)| \leq ch(t)h(t_0)^{-1}, \quad t \geq t_0 \in \mathbb{T}.$$

From Lemma 2.4 and by virtue of the boundedness of  $S(t)$  and  $S^{-1}(t)$ , there are positive constants  $c_1$  and  $c_2$  such that

$$\begin{aligned} |\Phi_B(t, t_0)| &\leq |S^{-1}(t)|[|\Phi_A(t, t_0)||S(t_0)| \\ &\quad + \int_{t_0}^t |\Phi_A(t, \sigma(s))||F(s)||\Phi_B(s, t_0)|\Delta s] \\ &\leq c_1c_2 \frac{h(t)}{h(t_0)} + c_1c_2 \int_{t_0}^t \frac{h(t)}{h(\sigma(s))}|F(s)||\Phi_B(s, t_0)|\Delta s, \end{aligned}$$

where  $\Phi_B(t, t_0)$  is a matrix exponential function for (2.2). Dividing by  $h(t)$  on both sides, we have

$$\frac{|\Phi_B(t, t_0)|}{h(t)} \leq \frac{c_1c_2}{h(t_0)} + c_1c_2 \int_{t_0}^t \frac{h(s)}{h(\sigma(s))}|F(s)| \frac{|\Phi_B(s, t_0)|}{h(s)} \Delta s.$$

In view of the Gronwall's inequality on time scale in [3], we have

$$\begin{aligned} \frac{|\Phi_B(t, t_0)|}{h(t)} &\leq \frac{c_1c_2}{h(t_0)} e_{p(t)}(t, t_0) = \frac{c_1c_2}{h(t_0)} \exp \left( \int_{t_0}^t \xi_{\mu(s)}(p(s)) \Delta s \right) \\ &= \begin{cases} \frac{c_1c_2}{h(t_0)} \exp \left( \int_{t_0}^t \frac{1}{\mu(s)} \text{Log}(1 + \mu(s)p(s)) \Delta s \right) & \text{if } \mu \neq 0 \\ \frac{c_1c_2}{h(t_0)} \exp \left( \int_{t_0}^t p(s) \Delta s \right) & \text{if } \mu = 0 \end{cases} \\ &\leq \frac{c_1c_2}{h(t_0)} \exp \left( \int_{t_0}^t p(s) \Delta s \right) \leq \frac{c_1c_2}{h(t_0)} \exp \left( \int_{t_0}^\infty p(s) \Delta s \right), \end{aligned}$$

where  $p(t) = c_1c_2 \frac{h(t)}{h(\sigma(t))}|F(t)|$  and the cylinder transformation  $\xi_\mu(z)$  is given by

$$\xi_\mu(z) = \begin{cases} \frac{1}{\mu} \text{Log}(1 + \mu z) & \text{if } \mu \neq 0 \\ z & \text{if } \mu = 0. \end{cases}$$

Thus we have

$$|\Phi_B(t, t_0)| \leq dh(t)h(t_0)^{-1}, \quad t \geq t_0,$$

where  $d \geq c_1c_2 \exp(\int_{t_0}^\infty p(s) \Delta s)$  is a positive constant. Hence (2.2) is an  $h$ -system by Lemma 3.3.

The converse holds by the same method. This completes the proof.  $\square$

REMARK 3.5. If  $h(t)$  is a positive bounded function on  $\mathbb{T}$ , then  $\frac{h(t)}{h(\sigma(t))}$  is not bounded in general. For example, see [6, Remark 3.1] for  $\mathbb{T} = \mathbb{Z}_+$ .

COROLLARY 3.6. Suppose that systems (2.1) and (2.2) are  $u_\infty$ -similar with bounded function  $\frac{h(t)}{h(\sigma(t))}$  on  $\mathbb{T}$ .

- (i) Then (2.1) is  $h$ -stable if and only if (2.2) is  $h$ -stable.
- (ii) When (2.1) is  $h$ -stable with  $h(t) = e_{-\lambda}(t, a_0)$  for some positive constant  $\lambda$  with  $-\lambda \in \mathcal{R}^+$  in Theorem 3.4, then (2.1) is uniformly exponentially stable if and only if (2.2) is uniformly exponentially stable.
- (iii) When (2.1) is  $h$ -stable with a constant function  $h$ , then (2.1) is uniformly stable if and only if (2.2) is also uniformly stable.

We can obtain the following results as special cases of Theorem 3.4.

REMARK 3.7. Suppose that systems (2.1) and (2.2) are  $u_\infty$ -similar with  $\int_{t_0}^{\infty} \frac{h(s)}{h(\sigma(s))} |F(s)| \Delta s < \infty$  for each  $t_0 \in \mathbb{T}$ .

- (i) When  $\mathbb{T} = \mathbb{R}$ , then (2.1) is  $h$ -stable if and only if (2.2) is also  $h$ -stable in [4, Theorem 2.2].
- (ii) If  $\mathbb{T} = \mathbb{Z}$ , then  $\int_{t_0}^{\infty} \frac{h(s)}{h(\sigma(s))} |F(s)| \Delta s < \infty$  becomes  $\sum_{s=t_0}^{\infty} \frac{h(s)}{h(s+1)} |F(s)| < \infty$  for each  $t_0 \in \mathbb{Z}$ . Thus (2.1) is  $h$ -stable if and only if (2.2) is also  $h$ -stable in [5, Theorem 3] and [8, Lemma 3.3].

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