JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **21**, No. 2, June 2008

RECURSIONS FOR TRACES OF SINGULAR MODULI

Chang Heon Kim*

Abstract. We will derive recursion formulas satisfied by the traces of singular moduli for the higher level modular function.

1. Introduction

Let $\mathfrak{H}$ be the complex upper half plane and let Γ be the full modular group $PSL_2(\mathbb{Z})$. Since Γ acts on $\mathfrak{H}$ by linear fractional transformations, we get the modular curve $\Gamma\backslash\mathfrak{H}^*$, as the projective closure of the smooth affine curve $\Gamma\backslash\mathfrak{H}$. Since the genus of $\Gamma\backslash\mathfrak{H}^*$ is zero, the function field of $\Gamma\backslash\mathfrak{H}^*$ is the rational function field $\mathbb{C}(j)$. Here j is the modular invariant which is uniquely characterized by $j(\infty) = \infty$, $j(\frac{-1+\sqrt{-3}}{2}) = 0$ and $j(\sqrt{-1}) = 1728$. The property $j(\tau+1) = j(\tau)$ implies that j admits a Fourier expansion with respect to $q = e^{2\pi i\tau}$ ($\tau \in \mathfrak{H}$), which is called a q-series (or q-expansion) as follows:

$$j(\tau) = q^{-1} + 744 + 196884q + \cdots .$$

"Singular values" or "singular moduli" is the classical name for the values assumed by the modular invariant $j(\tau)$ (or by other modular functions) when the argument is an imaginary quadratic irrationality. These values are algebraic numbers and have been studied intensively since the time of Kronecker and Weber. In [2], formulas for their norms and for the norms of their differences were obtained. In [3], a result for their traces and a number of generalizations were also obtained. Let d denote a positive integer congruent to 0 or 3 modulo 4. We denote by $\mathcal{Q}_d$ the set of positive definite binary quadratic forms $Q = [a,b,c] = aX^2 + bXY + cY^2$ ($a,b,c \in \mathbb{Z}$) of discriminant $-d$, with usual action of the modular group Γ. To each $Q \in \mathcal{Q}_d$, we associate its unique root

Received March 21, 2008; Accepted April 05, 2008.
2000 Mathematics Subject Classification: 11F03, 11F11, 11F30.
Key words and phrases: modular function, singular moduli, Jacobi form.
This work was supported by a research grant from Seoul Women's University (2007).

$\alpha_Q \in \mathfrak{H}$. Let $\mathbf{t}(d)$ be the (weighted) trace of a singular modulus of discriminant $-d$, that is,

$$\mathbf{t}(d) = \sum_{Q \in \mathcal{Q}_d/\Gamma} \frac{1}{|\bar{\Gamma}_Q|}(j(\alpha_Q) - 744).$$

Here $\bar{\Gamma}_Q = \{\gamma \in \bar{\Gamma} = PSL_2(\mathbb{Z}) \mid Q \circ \gamma = Q\}$. In addition we set $\mathbf{t}(-1) = -1, \mathbf{t}(0) = 2$ and $\mathbf{t}(d) = 0$ for $d < -1$ or $d \equiv 1, 2 \pmod 4$. Zagier's trace formula [3, Theorem 1] says that the series $\sum_{d \in \mathbb{Z}} \mathbf{t}(d) q^d$ ($q = e^{2\pi i \tau}, \tau \in \mathfrak{H}$) is a modular form of weight 3/2 on $\Gamma_0(4)$ ($= \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z}) : 4|c\}$), holomorphic in $\mathfrak{H}$ and meromorphic at cusps. Moreover he derived a recursion formula for $\mathbf{t}(d)$ (see [3, Theorem 2]).

Let $\Gamma_0(N)^*$ be the group generated by $\Gamma_0(N)$ ($= \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z}) : N|c\}$) and all Atkin-Lehner involutions W_e for $e||N$. Here $e||N$ denotes that $e|N$ and $(e, N/e) = 1$, and W_e can be represented by a matrix $\frac{1}{\sqrt{e}}\left(\begin{smallmatrix} ex & y \\ Nz & ew \end{smallmatrix}\right)$ with $\det W_e = 1$ and $x, y, z, w \in \mathbb{Z}$. There are only finitely many values of N for which $\Gamma_0(N)^*$ is of genus 0. In particular, if we let $\mathfrak{S}$ denote the set of prime values for such N, then

$$\mathfrak{S} = \{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 41, 47, 59, 71\}.$$

For each $p \in \mathfrak{S}$, let j_p^* be the corresponding Hauptmodul. Let d be an integer ≥ 0 such that $-d$ is congruent to a square modulo $4p$. We choose an integer $\beta \pmod{2p}$ with $\beta^2 \equiv -d \pmod{4p}$ and consider the set $\mathcal{Q}_{d,p,\beta} = \{[a, b, c] \in \mathcal{Q}_d \mid a \equiv 0 \pmod p,\ b \equiv \beta \pmod{2p}\}$ on which $\Gamma_0(p)$ acts. we define the trace $\mathbf{t}^{(p)}(d)$ by

$$\mathbf{t}^{(p)}(d) = \sum_{Q \in \mathcal{Q}_{d,p,\beta}/\Gamma_0(p)} \frac{1}{|\bar{\Gamma}_0(p)_Q|} j_p^*(\alpha_Q).$$

Here are some numerical examples when $p = 2$: first, we note that j_2^* can be expressed by means of Dedekind eta functions, that is,

$$j_2^*(\tau) = \left(\frac{\eta(\tau)}{\eta(2\tau)}\right)^{24} + 24 + 4096\left(\frac{\eta(2\tau)}{\eta(\tau)}\right)^{24}$$

where $\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1 - q^n)$. Then
$\mathbf{t}^{(2)}(4) = \frac{1}{2} j_2^*\left(\alpha_{[2,-2,1]}\right) = -52, \mathbf{t}^{(2)}(7) = j_2^*\left(\alpha_{[2,-1,1]}\right) = -23, \mathbf{t}^{(2)}(8) = j_2^*\left(\alpha_{[2,0,1]}\right) = 152, \mathbf{t}^{(2)}(12) = j_2^*\left(\alpha_{[2,2,2]}\right) + j_2^*\left(\alpha_{[4,2,1]}\right) = -496, \mathbf{t}^{(2)}(15) = j_2^*\left(\alpha_{[4,1,1]}\right) + j_2^*\left(\alpha_{[2,1,2]}\right) = -1$, $\mathbf{t}^{(2)}(16) = \frac{1}{2} j_2^*\left(\alpha_{[4,-4,2]}\right) + j_2^*\left(\alpha_{[2,0,2]}\right) + j_2^*\left(\alpha_{[4,0,1]}\right) = 1036$, etc.

In this article we will derive the following recursion formula for $\mathbf{t}^{(2)}(d)$:

THEOREM 1.1. *For all integers $n \geq 1$ we have the identities*

$$\boldsymbol{t}^{(2)}(8n-4) = -(120n-20)\sigma_3(n) + 42\sigma_5(n) - \sum_{3\leq r\leq\sqrt{8n+1}} \frac{r^4-r^2}{12}\boldsymbol{t}^{(2)}(8n-r^2)$$

$$\boldsymbol{t}^{(2)}(8n-1) = -240\sigma_3(n) - \sum_{2\leq r\leq\sqrt{8n+1}} r^2\boldsymbol{t}^{(2)}(8n-r^2)$$

$$\boldsymbol{t}^{(2)}(8n) = -2\sum_{1\leq r\leq\sqrt{8n+1}} \boldsymbol{t}^{(2)}(8n-r^2)$$

where $\sigma_k(n) = \sum_{\substack{d>0\\ d|n}} d^k$.

We note that the above recursion determines $\mathbf{t}^{(2)}(d)$ completely from the initial value $\mathbf{t}^{(2)}(-1) = -1$:
$\mathbf{t}^{(2)}(4) = -100\sigma_3(1) + 42\sigma_5(1) - \frac{3^4-3^2}{12}\mathbf{t}^{(2)}(-1) = -52$,
$\mathbf{t}^{(2)}(7) = -240\sigma_3(1) - 2^2\mathbf{t}^{(2)}(4) - 3^2\mathbf{t}^{(2)}(-1) = -23$,
$\mathbf{t}^{(2)}(8) = -2(\mathbf{t}^{(2)}(7) + \mathbf{t}^{(2)}(4) + \mathbf{t}^{(2)}(-1)) = 152$, etc.

2. Proof of Theorem 1.1

To prove Theorem 1.1 we first recall some basic facts on Jacobi forms. A *(holomorphic) Jacobi form of weight k and index p* is defined to be a holomorphic function $\phi : \mathfrak{H}\times\mathbb{C}\to\mathbb{C}$ satisfying the two transformation laws

$$\phi\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\right) = (c\tau+d)^k e^{2\pi i p\frac{cz^2}{c\tau+d}}\phi(\tau,z) \quad (\forall \left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right) \in SL_2(\mathbb{Z})),$$

$$\phi(\tau, z+\lambda\tau+\mu) = e^{-2\pi i p(\lambda^2\tau+2\lambda z)}\phi(\tau,z) \quad (\forall\ (\ \lambda,\mu\ ) \in \mathbb{Z}^2)$$

and having a Fourier expansion of the form

$$\phi(\tau,z) = \sum_{\substack{n,r\in\mathbb{Z}\\ 4pn-r^2\geq 0}} c(n,r)q^n\zeta^r \quad (q = e^{2\pi i\tau},\ \zeta = e^{2\pi i z}), \tag{2.1}$$

where the coefficient $c(n,r)$ depends only on $4pn-r^2$ if k is even and p is prime ([1] Theorem 2.2). The holomorphy condition at infinity is that $c(n,r)$ vanishes unless $4pn-r^2\geq 0$. If we relax the condition to merely requiring that $c(n,r)=0$ if $n<0$, we obtain the space of *weak Jacobi forms*, denoted $\tilde{J}_{k,p}$. Let $\tilde{J}_{*,*}$ be the ring of all weak Jacobi forms and $\tilde{J}_{ev,*}$ its even weight subring. Then $\tilde{J}_{ev,*}$ is the free polynomial algebra over $M_*(\Gamma)$ on two generators $a = \tilde{\phi}_{-2,1}(\tau, z) \in \tilde{J}_{-2,1}$ and $b = \tilde{\phi}_{0,1}(\tau, z) \in \tilde{J}_{0,1}$ (see [1, §9]). Here $M_*(\Gamma)$ denotes the ring of all modular forms on Γ, which is generated by Eisenstein series $E_4(\tau) = 1 + 240\sum_n \sigma_3(n)q^n$ and $E_6(\tau) = 1 - 504\sum_n \sigma_5(n)q^n$.

According to [3, §8] there is a Jacobi form $\phi^{(2)} \in \tilde{J}_{2,2}$ uniquely characterized by the requirement that it has Fourier coefficients $c(n, r) = B^{(2)}(8n - r^2)$ which depend only on the discriminant $8n - r^2$, with $B^{(2)}(0) = -2$, $B^{(2)}(-1) = 1$, $B^{(2)}(d) = 0$ for $d < -1$. In particular, the Fourier development of $\phi^{(2)}$ begins $(\zeta - 2 + \zeta^{-1}) + O(q)$. The representation of the form $\phi^{(2)}$ in terms of the generators a and b are $\phi^{(2)} = \frac{1}{12}a(E_4 b - E_6 a)$. Moreover Zagier's trace formula in higher level cases [3, Theorem 8] says that

$$\mathbf{t}^{(2)}(d) = -B^{(2)}(d). \tag{2.2}$$

Consider $\mathcal{D}_\nu : \tilde{J}_{k,m} \to M_{k+\nu}$ defined by

$$\mathcal{D}_0(\phi) = \sum_n (\sum_r c(n, r))q^n$$
$$\mathcal{D}_2(\phi) = \sum_n (\sum_r (kr^2 - 2nm)c(n, r))q^n$$
$$\mathcal{D}_4(\phi) = \sum_n (\sum_r ((k + 1)(k + 2)r^4 - 12(k + 1)r^2 nm + 12n^2m^2)c(n, r))q^n$$

(see [1, §3]). Now we fix $k = 2$, $m = 2$ and $\phi = \phi^{(2)}$. Since $M_2 = \{0\}$, $M_4 = \mathbb{C}E_4$ and $M_6 = \mathbb{C}E_6$, we obtain that $\mathcal{D}_0(\phi^{(2)}) = 0$, $\mathcal{D}_2(\phi^{(2)}) = c\cdot E_4$ and $\mathcal{D}_4(\phi^{(2)}) = c' \cdot E_6$ for some constants c and c'. Thus we have

$$\mathcal{D}_0(\phi^{(2)}) = \sum_n (\sum_{\substack{r\in\mathbb{Z} \\ r^2 \le 8n+1}} B^{(2)}(8n - r^2))q^n = 0 \tag{2.3}$$

$$\begin{aligned} \mathcal{D}_2(\phi^{(2)}) &= \sum_n (\sum_{\substack{r\in\mathbb{Z} \\ r^2 \le 8n+1}} (2r^2 - 4n)B^{(2)}(8n - r^2))q^n \\ &= c(1 + 240\sum_n \sigma_3(n)q^n) \end{aligned} \tag{2.4}$$

$$\begin{aligned} \mathcal{D}_4(\phi^{(2)}) &= \sum_n (\sum_{\substack{r\in\mathbb{Z} \\ r^2 \le 8n+1}} (12r^4 - 72r^2 n + 48n^2)B^{(2)}(8n - r^2))q^n \\ &= c'(1 - 504\sum_n \sigma_5(n)q^n) \end{aligned} \tag{2.5}$$

By comparing the constants terms in the above equations (2.4) and (2.5) we get $c = 2\cdot 2\cdot B^{(2)}(-1) = 4$ and $c' = 2\cdot 12\cdot B^{(2)}(-1) = 24$. Now if we compare the coefficients of q^n $(n \geq 1)$ in (2.3), (2.4) and (2.5), then we have for all $n \geq 1$,

$$B^{(2)}(8n) + 2 \sum_{1\leq r\leq \sqrt{8n+1}} B^{(2)}(8n - r^2) = 0 \tag{2.6}$$

$$\sum_{1\leq r\leq \sqrt{8n+1}} r^2 B^{(2)}(8n - r^2) = 240\sigma_3(n) \text{ by (2.3) and (2.4)} \tag{2.7}$$

$$\sum_{1\leq r\leq \sqrt{8n+1}} (r^4 - 6r^2 n) B^{(2)}(8n - r^2) = -504\sigma_5(n) \quad \text{by (2.3) and (2.5)} \tag{2.8}$$

We can simplify the equation (2.8) by making use of (2.7), that is,

$$\sum_{1\leq r\leq \sqrt{8n+1}} r^4 B^{(2)}(8n - r^2) = -504\sigma_5(n) + 1440n\sigma_3(n). \tag{2.9}$$

And then if we subtract (2.7) from (2.9) we obtain

$$\begin{aligned}\sum_{2\leq r\leq \sqrt{8n+1}} &(r^4 - r^2) B^{(2)}(8n - r^2) \\ &= -504\sigma_5(n) + (1440n - 240)\sigma_3(n).\end{aligned} \tag{2.10}$$

Now we can rewrite the equations (2.10), (2.7) and (2.6) as follows: for all $n \geq 1$,

$$\begin{aligned} B^{(2)}(8n-4) &= (120n - 20)\sigma_3(n) - 42\sigma_5(n) \\ &\qquad - \sum_{3\leq r\leq \sqrt{8n+1}} \frac{r^4 - r^2}{12} B^{(2)}(8n - r^2) \\ B^{(2)}(8n-1) &= 240\sigma_3(n) - \sum_{2\leq r\leq \sqrt{8n+1}} r^2 B^{(2)}(8n - r^2) \\ B^{(2)}(8n) &= -2 \sum_{1\leq r\leq \sqrt{8n+1}} B^{(2)}(8n - r^2). \end{aligned}$$

Finally by (2.2) the theorem is proved.

References

[1] M. Eichler and D. Zagier, *The Theory of Jacobi Forms*, Progress in Math. **55**, Bikhäuser-Verlag, Boston-Basel-Stuttgart, 1985.

[2] B. Gross and D. Zagier, *On singular moduli,* J. Reine Angew. Math., **355** (1985), 191-220.

[3] D. Zagier, *Traces of singular moduli,* Motives, polylogarithms and Hodge theory, Part I (Irvine, CA, 1998) (2002), Int. Press Lect. Ser., 3, I, Int. Press, Somerville, MA, 211-244.

*
Department of Mathematics
Seoul Women's University
Seoul 139-774, Republic of Korea
E-mail: chkim@swu.ac.kr