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SOME PROPERTIES OF THE SPACE OF FUZZY BOUNDED LINEAR OPERATORS

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Abstract. In this paper, we will show that $(CF(X,K),\chi_{\|\|\cdot\|\|})$ is a fuzzy Banach space using that the dual space X^* of a normed linear space X is a crisp Banach space. And for a normed linear space Y instead of a scalar field K , we obtain $(CF(X,Y),\rho^*)$ is a fuzzy Banach space under the some conditions.

1. Introduction and preliminaries

Katsaras and Liu [3] introduced the notions of fuzzy vector spaces and fuzzy topological vector spaces. These ideas were modified by Katsaras [1] and Katsaras defined the fuzzy norm on a vector space in [2]. In [4] Krishna and Sarma discussed the generation of a fuzzy vector topology from an ordinary vector topology on a vector space. Also Krishna and Sarma [5] observed the convergence of sequence of fuzzy points. Rhie, Choi and Kim [8] introduced the notion of the fuzzy α-Cauchy sequence of fuzzy points and the fuzzy completeness.

In this paper, we investigate a fuzzification of some theorems relative to a dual vector space.

Now, we explain some basic definitions and results from [1], [2], [3]. Let X be a nonempty set. A fuzzy set in X is an element of the set I^X of all functions from X into the unit interval I. χ_A denotes the characteristic function of the set A. If f is a function from X into Y and $\mu \in I^Y = \{\mu \mid \mu : Y \to [0,1]\}$, then $f^{-1}(\mu)$ is the fuzzy set in X defined by $f^{-1}(\mu) = \mu \circ f$. Also, for $\rho \in I^X, f(\rho)$ is the member

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of I^Y which is defined by

$$f(\rho)(y) = \begin{cases} \vee\{\rho(x) \mid x \in f^{-1}(y)\} & \text{if } f^{-1}(y) \neq \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

The symbols $\vee$ and $\wedge$ are used for the supremum and infimum of the family respectively. And we denote the support of $\mu \in I^X$ by

$$\text{supp}\mu = \{x \in X \mid \mu(x) > 0\}.$$

Let X be a vector space over K, where K denotes either the set of all the real or the complex numbers. Let $\mu_1, \mu_2, \cdots, \mu_n \in I^X$. The fuzzy set $\mu = \mu_1 \times \mu_2 \times \cdots \times \mu_n$ in X^n, is defined by

$$\mu(x_1, x_2, \cdots, x_n) = \mu_1(x_1) \wedge \mu_2(x_2) \wedge \cdots \wedge \mu_n(x_n).$$

DEFINITION 1.1. ([3]) $f : X^n \to X$, given by $f(x_1, x_2, \cdots, x_n) = x_1 + x_2 + \cdots + x_n$, then the fuzzy set $f(\mu)$ in X is called the *sum of the fuzzy sets* $\mu_1, \mu_2, \cdots, \mu_n$, and it is denoted by $\mu_1 + \mu_2 + \cdots + \mu_n$. That is

$$\begin{aligned}&(\mu_1 + \mu_2 + \cdots + \mu_n)(x) \\ &= \vee\{\mu_1(x_1) \wedge \mu_2(x_2) \wedge \cdots \wedge \mu_n(x_n) \mid x = x_1 + x_2 + \cdots + x_n\}.\end{aligned}$$

DEFINITION 1.2. ([3]) Let X be a vector space. For $\mu \in I^X$ and t a scalar, the fuzzy set $t\mu$ is the image of μ under the map $g : X \to X$, $g(x) = tx$, that is if $\mu \in I^X$ and $t \in K$, then

$$(t\mu)(x) = \begin{cases} \mu(x/t) & \text{if } t \neq 0 \\ 0 & \text{if } t = 0 \text{ and } x \neq 0 \\ \vee\{\mu(y) \mid y \in X\} & \text{if } t = 0 \text{ and } x = 0. \end{cases}$$

DEFINITION 1.3. ([3]) A subfamily τ of I^X is said to be a *fuzzy topology* on a set X if,

1. τ contains every constant fuzzy set in X,
2. if $\mu_1, \mu_2 \in \tau$, then $\mu_1 \wedge \mu_2 \in \tau$,
3. if for each $\{\mu_i\}_i \subset \tau$, then $\vee_i \mu_i \in \tau$.

A fuzzy topological space is a set X equipped with a fuzzy topology τ, denote (X, τ). The elements of τ are called the *open fuzzy sets* in X.

DEFINITION 1.4. ([1]) A map f from a fuzzy topological space X to a fuzzy topological space Y, is said to be *fuzzy continuous* if $f^{-1}(\mu)$ is fuzzy open in X for each open fuzzy set μ in Y.

Definition 1.5. ([2]) A *fuzzy linear topology* on a vector space X over K is a fuzzy topology on X such that the two mappings

$$\begin{array}{lll} + : & X \times X \to X, & (x, y) \to x + y \\ \cdot : & K \times X \to X, & (t, x) \to tx \end{array}$$

are continuous when K has the fuzzy usual topology and $K \times X$ and $X \times X$ have the corresponding product fuzzy topologies. A linear space with a fuzzy linear topology is called a *fuzzy topological vector space.*

Definition 1.6. ([2]) $\mu \in I^X$ is said to be

1. *convex* if $t\mu + (1-t)\mu \subseteq \mu$ for each $t \in [0, 1]$
2. *balanced* if $t\mu \subseteq \mu$ for each $t \in K$ with $|\, t \,| \leq 1$
3. *absolutely convex* if μ is convex and balanced
4. *absorbing* if $\vee\{t\mu(x) \mid t > 0\} = 1$ for all $x \in X$.

Definition 1.7. ([2]) *fuzzy seminorm* on X is a fuzzy set ρ in X which is absolutely convex and absorbing. If in addition $\wedge\{(t\rho)(x) \mid t > 0\} = 0$ for $x \neq 0$, then ρ is called a *fuzzy norm.*

Theorem 1.8. *([1]) If ρ is a fuzzy seminorm on X, then the family $B_\rho = \{\theta \wedge (t\rho) \mid 0 < \theta \leq 1,\ t > 0\}$ is a base at zero for a fuzzy linear topology τ_ρ.*

Definition 1.9. ([2]) Let ρ be a fuzzy seminorm on a linear space. The fuzzy topology τ_ρ in Theorem 1.8 is called the *fuzzy topology induced by the fuzzy seminorm* ρ. And a linear space equipped with a fuzzy seminorm (resp. fuzzy norm) is called a *fuzzy seminormed* (resp. *fuzzy normed*) *linear space.*

Theorem 1.10. *([1]) The fuzzy seminorms ρ_1, ρ_2 on a linear space X are equivalent if and only if for each $\theta \in (0, 1)$, there exists $t > 0$ such that $\theta \wedge \rho_1(tx) \leq \rho_2(x)$ and $\theta \wedge \rho_2(tx) \leq \rho_1(x)$ for all $x \in X$.*

Definition 1.11. ([2]) A fuzzy set $\mu \in I^X$ is called a *fuzzy point* iff

$$\mu(z) = \begin{cases} \alpha & \text{if } z = x, \\ 0 & \text{otherwise,} \end{cases}$$

where $\alpha \in (0, 1)$. We denote this fuzzy point with support x and value α by (x, α).

2. Main theorem

DEFINITION 2.1. [8] Let $\alpha \in (0,1)$. A sequence of fuzzy points $\{\mu_n = (x_n, \alpha_n)\}$ is said to be a *fuzzy α-Cauchy sequence* in a fuzzy normed linear space (X, ρ) if for each neighborhood N of 0 with $N(0) > \alpha$, there exists a positive integer M such that $n, m \geq M$ implies $\mu_n - \mu_m = (x_n - x_m, \alpha_n \wedge \alpha_m) \leq N$. A fuzzy normed linear space (X, ρ) is said to be *fuzzy α-complete* if every fuzzy α-Cauchy sequence $\{\mu_n\}$ converges to a fuzzy point $\mu = (x, \alpha)$ (refer to Definition 2.13 of [5]). (X, ρ) is said to be *fuzzy complete* if it is fuzzy α-complete for every $\alpha \in (0,1)$. A fuzzy complete fuzzy normed linear space is said to be a *fuzzy Banach space*.

DEFINITION 2.2. [1] If ρ is a fuzzy seminorm on X, then for every $\epsilon \in (0,1)$, $P_\epsilon : X \to R_+$ is defined by

$$P_\epsilon(x) = \wedge\{t > 0 \mid t\rho(x) > \epsilon\}$$

and for every $x \in X$, $P_{\alpha^-} : X \to R_+$ is also defined by

$$P_{\alpha^-}(x) = \vee\{P_\epsilon(x) \mid \epsilon < \alpha\}.$$

THEOREM 2.3. [1] *The P_ϵ in Definition 2.2 is a seminorm on X. Further P_ϵ is a norm on X for each $\epsilon \in (0,1)$ if and only if ρ is a fuzzy norm on X.*

DEFINITION 2.4. [5] Let (X, ρ_1), (Y, ρ_2) be fuzzy normed linear spaces and $CF(X,Y)$ be the linear space of all fuzzy continuous linear maps from (X, ρ_1) to (Y, ρ_2). For each $\theta \in (0,1)$, $t_\theta : CF(X,Y) \to R_+$ is defined by

$$t_\theta(f) = \wedge\{s > 0 \mid \rho_2(f(x)) \geq \theta \wedge \rho_1(sx) \text{ for all } x \in X\}.$$

We write $t_\theta(f) = t(\theta, f)$. And the fuzzy norm $\rho^* : CF(X,Y) \to [0,1]$ is defined by $\rho^*(f) = \vee_{\theta \in (0,1)}\{\theta \wedge 1/[t(\theta, f)]\}$, for any $f \in CF(X,Y)$.

LEMMA 2.5. *Let $(X, \|\cdot\|)$ be a normed linear space. If $\rho = \chi_B$, where B is the closed unit ball of X, then for each $\epsilon \in (0,1)$, $P_\epsilon(x) = \| x \|$ for all $x \in X$.*

Proof. For all $x \in X$, $\epsilon \in (0,1)$,

$$\begin{aligned} P_\epsilon(x) &= \wedge\{s > 0 \mid s\rho(x) > \epsilon\} \\ &= \wedge\{s > 0 \mid \rho(x/s) > \epsilon\} \\ &= \wedge\{s > 0 \mid \rho(x/s) = 1\} && \text{as} \quad \rho = \chi_B \\ &= \wedge\{s > 0 \mid \| x/s \| \leq 1\} && \text{as} \quad x/s \in B \\ &= \wedge\{s > 0 \mid \| x \| \leq s\} \\ &= \| x \| . \end{aligned}$$

□

THEOREM 2.6. *Let* $(X, \| \cdot \|)$ *be a normed linear space over* $(K, | \cdot |)$. *Then* $(CF(X,K), \chi_{\|| \cdot \||})$*is fuzzy Banach space, where* $\rho_1 = \chi_{\| \cdot \|}$, $\rho_2 = \chi_{| \cdot |}$ *and* $\||\, x^* \,\||= \vee\{| \, x^*(x) \, | \mid P^1_\epsilon(x) = 1, \; x \in X\}$

Proof. Since we have that for each $\epsilon \in (0,1)$,

$$\begin{aligned} P^1_\epsilon(x) &= \| x \| \quad \text{for each } x \in X \\ P^2_\epsilon(y) &= | \, y \, | \quad \text{for each } y \in K. \end{aligned}$$

Since $(X, \| \cdot \|)$, $(K, | \cdot |)$ are normed linear space, (X, ρ_1), (K, ρ_2) are fuzzy normed linear space.Thus

$$\begin{aligned} X^* &= \{x^* | \; x^* : (X, \| \cdot \|) \to (K, | \cdot |) \;\; \text{is continuous and linear } \} \\ &= \{x^* | \; x^* : (X, \rho_1) \to (K, \rho_2) \;\; \text{is fuzzy continuous and linear } \} \\ &= CF(X,K) \end{aligned}$$

Since $(X^*, \|| \cdot \||)$ is Banach space,where $\||\, x^* \,\||= \vee\{| \, x^*(x) \, | \mid x \in X, P^1_\epsilon(x) = 1\}$, $(X^*, \chi_{\|| \cdot \||})$ is fuzzy complete. Consequently $(CF(X,K)$, $\chi_{\|| \cdot \||})$ is fuzzy complete. This completes the proof. □

DEFINITION 2.7. [2] Two fuzzy seminorms ρ_1, ρ_2 on X are said to be *equivalent* if $\tau_{\rho_1} = \tau_{\rho_2}$.

PROPOSITION 2.8. [8] *Let* $(X, \| \cdot \|)$ *be a normed linear space. If* ρ *be a lower semi-continuous fuzzy norm on* X, *and have the bounded support:* $\{x \in X | \; \rho(x) > 0\}$ *is bounded, then* ρ *is equivalent to the fuzzy norm* χ_B *where* B *is the closed unit ball of* X.

THEOREM 2.9. *Let* $(X, \| \cdot \|_1)$ *and* $(Y, \| \cdot \|_2)$ *be two normed linear spaces over the field* K. *If* $f : (X, \| \cdot \|_1) \to (Y, \| \cdot \|_2)$ *is continuous and linear . Then* $f : (X, \rho_1) \to (Y, \rho_2)$ *is fuzzy continuous, where* $\rho_1 = \chi_{\| \cdot \|_1}$ *and* $\rho_2 = \chi_{\| \cdot \|_2}$

Proof. Let $\theta \in (0,1)$. We have to show that there exists $t = t(\theta) > 0$ such that $\rho_2(f(x)) \geq \theta \wedge \rho_1(tx)$ for all $x \in X$. Equivalently $\| tx \|_1 \leq 1$ implies $\| f(x) \|_2 \leq 1$. Let $t_\theta(f) = \wedge\{s > 0| \ \| f(x) \|_2 \leq s \| x \|_1 , x \in X\} = \||\ f \ \||$. Now, suppose that $\| f(x) \|_2 > 1$. Then since $\| f(x) \|_2 \leq \||\ f \ \|| \cdot \| x \|_1$ for each $x \in X$, $1 < \| f(x) \|_2 \leq \||\ f \ \|| \cdot \| x \|_1$ for each $x \in X$. Thus $\| tx \|_1 = |t| \cdot \| x \|_1 = \||\ f \ \|| \cdot \| x \|_1 > 1$. □

THEOREM 2.10. *Let $(X, \| \cdot \|_1)$ and $(Y, \| \cdot \|_2)$ be two normed linear spaces. If (Y, ρ_2) be a fuzzy complete, where $\rho_1 = \chi_{\|\cdot\|_1}$, $\rho_2 = \chi_{\|\cdot\|_2}$ and ρ_2 is lower semi continuous and has the bounded support, then $(CF(X,Y), \rho^*)$ is fuzzy complete, where $\rho^* = \chi_{\||\cdot\||}$, $\||\ x^* \ \|| = \vee\{P^2_\epsilon(x^*(x)) \mid P^1_\epsilon(x) = 1, x \in X\}$.*

Proof. From [5] and above Theorem 2.9, we have that

$$\begin{aligned}
&CF(X,Y) = \{f | f : (X, \rho_1) \to (Y, \rho_2) \ \text{ is fuzzy continuous and linear }\} \\
&= \{f | f : (X, P^1_\epsilon) \to (Y, P^2_\epsilon) \text{ is continuous and linear for each } \epsilon \in (0,1)\} \\
&= \{f | f : (X, \| \cdot \|_1) \to (Y, \| \cdot \|_2) \ \text{ is continuous and linear }\} \\
&= L(X,Y)
\end{aligned}$$

,where

$$\begin{aligned}
\||\ T \ \|| &= \vee\{P^2_\epsilon(T(x)) | \ P^1_\epsilon(x) = 1, x \in X\} \\
&= \vee\{\| T(x) \|_2 \ \mid P^1_\epsilon(x) = 1, x \in X\}, \quad T \in L(X,Y).
\end{aligned}$$

And since (Y, ρ_2) fuzzy complete, for each $\alpha \in (0,1)$, α-Cauchy sequence $(T_n(x), \alpha_n)$ converges to $(T(x), \alpha)$. Thus $T_n(x)$ converges to $T(x)$. Now we will show that $(CF(X,Y), \rho^*)$ is fuzzy complete. Let $\{T_n\} \subseteq CF(X,Y)$ is a fuzzy α-Cauchy sequence for each $\alpha \in (0,1)$, that is for each $t > 0$, there exists a positive integer M such that $n, m \geq M$ implies $\alpha_n \wedge \alpha_m \leq \alpha$ and $P^-_{(\alpha_n \wedge \alpha_m)}(x_n - x_m) < t$.Then $T_n : (X, \rho_1) \to (Y, \rho_2)$ is a fuzzy continuous and linear. Thus, by Theorem 4.9 [5] for each $\epsilon \in (0,1)$, $T_n : (X, P^1_\epsilon) \to (Y, P^2_\epsilon)$ is a crisp continuous and linear. Hence T_n is a bounded and linear. And since $T_n - T_m$ is bounded, it deduce that

$$\begin{aligned}
P^2_\epsilon(T_n(x) - T_m(x)) &\leq \ \||\ T_n - T_m \ \||\ P^1_\epsilon(x) \\
&= \ P_{(\alpha_n \wedge \alpha_m)^-}(T_n - T_m) P^1_\epsilon(x) \\
&< \ t P^1_\epsilon(x)
\end{aligned}$$

Therefore, $\{T_n(x)\}$ is a crisp Cauchy sequence in (Y, P^2_ϵ).

Since $\rho_2 = \chi_{\|\cdot\|_2}$ is lower semi continuous and has the bounded support, $(Y, \| \cdot \|_2)$ is a crisp complete. It mean that $(L(X,Y), \||\cdot\||)$ is a crisp Banach space by Theorem 3.2.2 [7]. Hence $(CF(X,Y), \||\cdot\||)$ is a crisp Banach space. Consequently, $(CF(X,Y), \chi_{|||\cdot|||})$ is fuzzy Banach space. This completes the proof. □

THEOREM 2.11. *Let* $(X, \| \cdot \|_1)$ *and* $(Y, \| \cdot \|_2)$*be two normed linear spaces over the field* K, $X \neq \{\theta\}$. *If* $(CF(X,Y), \rho^*)$ *be fuzzy Banach space, where* $\rho^* = \chi_{|||\cdot|||}$ *is lower semi continuous and has the bounded support,* $||| f ||| = \vee\{\| f(x) \|_2 \mid x \in X, \| x \|_1 = 1\}$. *Then* $(Y, \chi_{\|\cdot\|_2})$ *is fuzzy Banach space.*

Proof. Since $(CF(X,Y), \rho^*)$ be fuzzy Banach space and ρ^* is lower semi continuous and has the bounded support, $(CF(X,Y), ||| \cdot |||)$ is crisp Banach space. It follows that $(Y, \| \cdot \|_2)$ is crisp Banach space from [7]. Consequently, $(Y, \chi_{\|\cdot\|_2})$ is fuzzy Banach space. □

References

[1] A. K. Katsaras, Fuzzy topological vector space I, Fuzzy sets and systems **6** (1981), 85-95.

[2] A. K. Katsaras, Fuzzy topological vector space II, Fuzzy sets and systems **12** (1984), 143- 154.

[3] A. K. Katsaras and D. B. Liu, Fuzzy vector spaces and fuzzy topological vector spaces, J. Math. Anal. Appl. **58** (1977), 135-146.

[4] S. V. Krishna and K. K. M. Sarma, Fuzzy topological vector spaces - topological generation and normability, Fuzzy sets and systems **41** (1991), 89-99.

[5] S. V. Krishna and K. K. M. Sarma, Fuzzy continuity of linear maps on vector spaces, Fuzzy sets and systems **45** (1992), 341-354.

[6] S. V. Krishna and K. K. M. Sarma, Seperation of fuzzy normed linear spaces, Fuzzy sets and systems **63** (1994), 207-217.

[7] R. Larsen, Functional analysis, Marcel Dekker, Inc. New york, 1973.

[8] G. S. Rhie and B. M. Choi and D. S. Kim, On the completeness of fuzzy normed linear spaces, Math. Japonica **45**, (1997), no. 1, 33-37.

[9] G. S. Rhie and I. A. Hwang, On the fuzzy Hahn-Banach Theorem-an analytic form, Fuzzy sets systems **108** (1999), 117-121.

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