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THE GENERALIZED HYERS-ULAM-RASSIAS STABILITY OF A CUBIC FUNCTIONAL EQUATION

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Abstract. In this paper, we obtain the general solution, the generalized Hyers-Ulam-Rassias stability, and the stability by using the alternative fixed point for a cubic functional equation

$4f(x+my)+4f(x-my)+m^2f(2x) = 8f(x)+4m^2f(x+y)+4m^2f(x-y)$

for a positive integer $m \geq 2$.

1. Introduction

The study of stability problems for functional equations is related to the following question originated by Ulam [14] concerning the stability of group homomorphisms: *Let G_1 be a group and let G_2 be a metric group with the metric $d(\cdot,\cdot)$. Given $\varepsilon > 0$ does there exist a $\delta > 0$ such that if a mapping $h : G_1 \to G_2$ satisfies the inequality*

$$d(h(xy),\, h(x)h(y)) < \delta$$

for all $x, y \in G_1$, then a homomorphism $H : G_1 \to G_2$ exists with $d(h(x),\, H(x)) < \varepsilon$ for all $x \in G_1$?

The first partial solution to Ulam's question was provided by D. H. Hyers [5]. Thirty seven years after Hyers's Theorem, Th. M. Rassias in his paper provided a remarkable generalization of Hyers's result by allowing for the first time in the subject of functional equations and inequalities the Cauchy difference to be unbounded; see [8]. This fact rekindled interest of several mathematicians worldwide in the study of several important functional equations of several variables. Găvruta [4] following Rassias's approach for the unbounded Cauchy difference provided a further generalization.

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The quadratic function $f(x) = cx^2\,(c \in \mathbb{R})$ satisfies the functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y)\,. \tag{1.1}$$

This question is called the quadratic functional equation, and every solution of the equation (1.1) is called a quadratic function. In fact, a function $f : X \to Y$ is a solution of the equation (1.1) if and only if there exists a bilinear function $B : X \times X \to Y$ such that $f(x) = B(x,x)$ for all $x \in X$.

A Hyers-Ulam stability theorem for the quadratic functional equation (1.1) was first proved by Skof [13] for functions $f : X \to Y$, where X is a normed space and Y is a Banach space. Cholewa [1] noticed that the theorem of Skof is still true if the relevant domain X is replaced by an abelian group. In [3], Czerwik proved the Hyers-Ulam-Rassias stability of the quadratic functional equation. Several functional equations have been investigated; see [9], [10], and [11].

The cubic function $f(x) = cx^3\,(c \in \mathbb{R})$ satisfies the functional equation

$$f(2x+y) + f(2x-y) = 2f(x+y) + 2f(x-y) + 12f(x)\,. \tag{1.2}$$

The equation (1.2) was solved by Jun and Kim [6]. Similar to a quadratic functional equation, actually, they proved that a function $f : X \to Y$ is a solution of the equation (1.2) if and only if there exists a function $F : X \times X \times X \to Y$ such that $f(x) = F(x,x,x)$ for all $x \in X$, and F is symmetric for each fixed one variable and is additive for fixed two variables; see [6]. We promise that by a cubic function we mean every solution of the equation (1.2) is called a cubic function. Also, the equation (1.2) is equivalent to the following equation (see [2, Lemma 2.1]);

$$f(x+2y) + f(x-2y) + f(2x) = 2f(x) + 4f(x+y) + 4f(x-y)\,. \tag{1.3}$$

In this paper, we will investigate the generalized Hyers-Ulam-Rassias stability and the stability by using the alternative fixed point for a cubic functional equation as follows:

$$\begin{aligned} &4f(x+my) + 4f(x-my) + m^2 f(2x) \\ &= 8f(x) + 4m^2 f(x+y) + 4m^2 f(x-y) \end{aligned} \tag{1.4}$$

for all $x, y \in X$, where $m \geq 2$ is an integer number.

2. Cubic functional equation

LEMMA 2.1. *Let X and Y be real vector spaces. A function $f: X \to Y$ satisfies the functional equation (1.4) if and only if f is cubic. Therefore, every solution of functional equations (1.4) is also a cubic function.*

Proof. Suppose f satisfies the equation (1.3). It is easy to show that $f(2x) = 8f(x)$, for all $x \in X$. Hence the equation (1.3) is equivalent to

$$f(x+2y) + f(x-2y) + 6f(x) = 4f(x+y) + 4f(x-y).$$

By letting $x = x+y$ and $x = x-y$ respectively, we have

$$\begin{aligned} (2.1)\quad f(x+3y) + f(x-y) + 6f(x+y) &= 4f(x+2y) + 4f(x) \\ (2.2)\quad f(x+y) + f(x-3y) + 6f(x-y) &= 4f(x) + 4f(x-2y). \end{aligned}$$

Adding (2.1) to (2.2) and using the previous equation, we have

$$\begin{aligned} &f(x+3y) + f(x-3y) + 7f(x+y) + 7f(x-y) \\ = \ &8f(x) + 4f(x+2y) + 4f(x-2y) \\ = \ &16f(x+y) + 16f(x+y) - 16f(x). \end{aligned}$$

By multiplying by 4 and using $f(2x) = 8f(x)$, we obtain

$$\begin{aligned} &4f(x+3y) + 4f(x-3y) + 3^2 f(2x) \\ &= 8f(x) + 4 \cdot 3^2 f(x+y) + 4 \cdot 3^2 f(x-y). \end{aligned}$$

By using above method and induction, we infer that

$$4f(x+my)+4f(x-my)+m^2 f(2x) = 8f(x)+4m^2 f(x+y)+4m^2 f(x-y),$$

for all $x, y \in X$ and each integer $m \geq 2$.

Conversely, suppose that f satisfies the equation (1.4) for each integer $m \geq 2$. By letting $m = 2$,

$$4f(x+2y) + 4f(x-2y) + 4f(2x) = 8f(x) + 16f(x+y) + 16f(x-y).$$

Also, it satisfies that $f(2x) = 8f(x)$, for all $x \in X$. Enough to check the case where $m \geq 3$ is an integer number. By letting $x = x+y$ and $x = x-y$ respectively, we obtain

$$\begin{aligned} &4f(x+(m+1)y) + 4f(x-(m-1)y) + m^2 f(2(x+y)) \\ = \ &8f(x+y) + 4m^2 f(x+2y) + 4m^2 f(x), \\ &4f(x+(m-1)y) + 4f(x-(m+1)y) + m^2 f(2(x-y)) \\ = \ &8f(x-y) + 4m^2 f(x) + 4m^2 f(x-2y). \end{aligned}$$

By adding two equations, we have

$$\begin{aligned}
& 4f(x+(m+1)y)+4f(x-(m+1)y)+4f(x+(m-1)y) \\
& +4f(x-(m-1)y)+m^2f(2(x+y))+m^2f(2(x-y)) \\
= \; & 8f(x+y)+8f(x-y)+4m^2f(x+2y)+4m^2f(x-2y)+8m^2f(x)\,.
\end{aligned}$$

Now, by using the cases where $m=2$ and $m=m+1$,

$$\begin{aligned}
& 4(m+1)^2f(x+y)+4(m+1)^2f(x-y)+8f(x)-(m+1)^2f(2x) \\
& +4f(x+(m-1)y)+4f(x-(m-1)y) \\
& +m^2f(2(x-y))+m^2f(2(x+y)) \\
= \; & 8f(x-y)+8f(x+y)+16m^2f(x+y)+16m^2f(x-y) \\
& -24m^2f(x)+8m^2f(x)\,.
\end{aligned}$$

Hence we have

$$\begin{aligned}
& 4f(x+(m-1)y)+4f(x-(m-1)y)+(m-1)^2f(2x) \\
= \; & 8f(x)+4(m-1)^2f(x+y)+4(m-1)^2f(x-y)\,,
\end{aligned}$$

for all $x, y \in X$. By using above method and induction, we can reduce to the equation (1.3) for any integer $m \geq 2$. Thus f is cubic. □

3. Stability

Throughout in this section, let X be a normed vector space with norm $\| \cdot \|$ and Y a Banach space with norm $\| \cdot \|$. For the given mapping $f: X \to Y$, we define

$$Df(x,y) := 4f(x+my)+4f(x-my)+m^2f(2x) \\ -8f(x)-4m^2f(x+y)-4m^2f(x-y)\,, \tag{3.1}$$

for all $x, y \in X$ and each integer $m \geq 2$.

THEOREM 3.1. *Let $f: X \to Y$ be a mapping for which there exists a function $\phi: X \times X \to [0, \infty)$ such that*

$$\widetilde{\phi}(x,y) := \sum_{j=0}^{\infty} (\frac{1}{8})^j \phi(2^j x, 2^j y) < \infty\,, \tag{3.2}$$

$$\| Df(x,y) \| \leq \phi(x,y)\,, \tag{3.3}$$

for all $x, y \in X$. Then there exists a unique cubic mapping $C : X \to Y$ such that

$$\| f(x) - C(x) \| \leq \frac{1}{8m^2} \widetilde{\phi}(x, 0) \,, \tag{3.4}$$

for all $x \in X$, and each integer $m \geq 2$.

Proof. By letting $y = 0$ in the equation (3.3), we have

$$\| f(x) - \frac{1}{8} f(2x) \| \leq \frac{1}{8m^2} \phi(x, 0) \,, \tag{3.5}$$

for all $x \in X$. Replacing x by $2x$ in the equation (3.5), we have

$$\| f(2x) - \frac{1}{8} f(2^2 x) \| \leq \frac{1}{8m^2} \phi(2x, 0) \,, \tag{3.6}$$

for all $x \in X$. Now, combining equations (3.5) and (3.6), we get

$$\| f(x) - (\frac{1}{8})^2 f(2^2 x) \| \leq \frac{1}{8m^2} \Big(\phi(x, 0) + \frac{1}{8} \phi(2x, 0) \Big) \,,$$

for all $x \in X$.

Continue this way, we may have

$$\| f(x) - (\frac{1}{8})^n f(2^n x) \| \leq \frac{1}{8m^2} \sum_{j=0}^{n-1} (\frac{1}{8})^j \phi(2^j x, 0) \,, \tag{3.7}$$

for all positive integer n and all $x \in X$.

For any positive integer s , dividing the equation (3.7) by 8^s and then substituting x by $2^s x$, we have

$$\begin{aligned} &(\frac{1}{8})^s \| f(2^s x) - (\frac{1}{8})^n f(2^{s+n} x) \| \\ &\qquad \leq (\frac{1}{8})^s \cdot \frac{1}{8m^2} \sum_{j=0}^{n-1} (\frac{1}{8})^j \phi(2^{s+j} x, 0) \,, \end{aligned}$$

for all $x \in X$.

By taking $s \to \infty$, we may conclude that $\{(\frac{1}{8})^n f(2^n x)\}$ is a Cauchy sequence in a Banach space Y . This implies that the sequence $\{(\frac{1}{8})^n f(2^n x)\}$ converges. Hence we can define a function $C : X \to Y$ by

$$C(x) = \lim_{n \to \infty} (\frac{1}{8})^n f(2^n x) \,,$$

for all $x \in X$. Then

$$\begin{aligned}\| DC(x,y) \| &= \lim_{n\to\infty} (\frac{1}{8})^n \| Df(2^n x, 2^n y) \| \\ &\le \lim_{n\to\infty} (\frac{1}{8})^n \phi(2^n x, 2^n y) \\ &= 0\,,\end{aligned}$$

for all $x, y \in X$. That is, $DC(x,y) = 0$. By Lemma 2.1, the function $C : X \to Y$ is cubic. It only remains to show that the function C is unique. Let $C' : X \to Y$ be another cubic function satisfying the equation (3.4). Then

$$\begin{aligned}\| C(x) - C'(x) \| &= (\frac{1}{8})^n \| C(2^n x) - C'(2^n x) \| \\ &\le (\frac{1}{8})^n (\| C(2^n x) - f(2^n x) \| + \| f(2^n x) - C'(2^n x) \|) \\ &\le (\frac{1}{8})^n \frac{1}{8} \widetilde{\phi}(2^n x, 0)\,,\end{aligned}$$

for all $x \in X$. As $n \to \infty$, we can conclude that $C(x) = C'(x)$, for all $x \in X$; that is, C is unique. □

THEOREM 3.2. *Let $f : X \to Y$ be a mapping for which there exists a function $\phi : X \times X \to [0, \infty)$ such that*

$$\widetilde{\phi}(x,y) := \sum_{j=1}^{\infty} 8^j \phi(2^{-j} x, 2^{-j} y) < \infty\,, \tag{3.8}$$

$$\| Df(x,y) \| \le \phi(x,y)\,, \tag{3.9}$$

for all $x, y \in X$. Then there exists a unique cubic mapping $C : X \to Y$ such that

$$\| f(x) - C(x) \| \le \frac{1}{m^2} \widetilde{\phi}(x, 0)\,, \tag{3.10}$$

for all $x \in X$, and each integer $m \ge 2$.

Proof. If x is replaced by $\frac{1}{2}x$ in the equation (3.5) in the proof of Theorem 3.1, we have

$$\Big| | f(x) - 8f(\frac{1}{2}x) \Big| | \le \frac{1}{m^2} \phi(\frac{1}{2}x, 0)\,,$$

for all $x \in X$. The remains of the proof are similar to the proof of Theorem 3.1. □

4. Stability using alternative fixed point

In this section, we will investigate the stability of the given cubic functional equation (3.1) using the alternative fixed point. Before proceeding the proof, we will state the theorem, the alternative of fixed point.

THEOREM 4.1 (The alternative of fixed point [7], [12]). *Suppose that we are given a complete generalized metric space* (Ω, d) *and a strictly contractive mapping* $T : \Omega \to \Omega$ *with Lipschitz constant* L. *Then for each given* $x \in \Omega$, *either*

$$d(T^n x,\, T^{n+1}x) = \infty \text{ for all } n \geq 0\,,$$

or there exists a natural number n_0 *such that*

1. $d(T^n x,\, T^{n+1}x) < \infty$ *for all* $n \geq n_0$;
2. *The sequence* $(T^n x)$ *is convergent to a fixed point* y^* *of* T;
3. y^* *is the unique fixed point of* T *in the set*

$$\triangle = \{y \in \Omega | d(T^{n_0}x, y) < \infty\}\,;$$

4. $d(y, y^*) \leq \frac{1}{1-L}\, d(y, Ty)$ *for all* $y \in \triangle$.

Now, let $\phi : X \times X \to [0, \infty)$ be a function such that

$$\lim_{r\to\infty} \frac{\phi(\lambda_i^r x, \lambda_i^r y)}{\lambda_i^{3r}} = 0\,,$$

for all $x, y \in X$, where $\lambda_i = 2$ if $i = 0$ and $\lambda_i = \frac{1}{2}$ if $i = 1$.

THEOREM 4.2. *Suppose that a function* $f : X \to Y$ *satisfies the functional inequality*

$$\| Df(x,y) \| \leq \phi(x,y)\,, \tag{4.1}$$

for all $x, y \in X$. *If there exists* $L = L(i) < 1$ *such that the function*

$$x \mapsto \psi(x) = \phi(\frac{1}{2}x, 0) \tag{4.2}$$

has the property

$$\psi(x) \leq L \cdot \lambda_i^3 \cdot \psi(\frac{x}{\lambda_i})\,, \tag{4.3}$$

for all $x \in X$, *then there exists a unique cubic function* $C : X \to Y$ *such that the inequality*

$$\| f(x) - C(x) \| \leq \frac{L^{1-i}}{1-L}\, \psi(x) \tag{4.4}$$

holds for all $x \in X$.

Proof. Consider the set

$$\Omega = \{g | g : X \to Y\}$$

and introduce the generalized metric on Ω,

$$d(g,\, h) = d_\psi(g, h) = \inf\{K \in (0, \infty) | \ \| g(x) - h(x) \| \leq K\psi(x) \,, x \in X\} \,.$$

It is easy to show that (Ω, d) is complete. Now we define a function $T : \Omega \to \Omega$ by

$$Tg(x) = \frac{1}{\lambda_i^3}\, g(\lambda_i x) \,,$$

for all $x \in X$. Note that for all $g, h \in \Omega$,

$$\begin{aligned} d(g, h) < K \quad &\Rightarrow \quad \| g(x) - h(x) \| \leq K\psi(x) \,, \text{ for all } x \in X \,, \\ &\Rightarrow \quad \| \frac{1}{\lambda_i^3} g(\lambda_i x) - \frac{1}{\lambda_i^3} h(\lambda_i x) \| \leq \frac{1}{\lambda_i^3} K\psi(\lambda_i x) \,, \text{ for all } x \in X \,, \\ &\Rightarrow \quad \| \frac{1}{\lambda_i^3} g(\lambda_i x) - \frac{1}{\lambda_i^3} h(\lambda_i x) \| \leq LK\psi(x) \,, \text{ for all } x \in X \,, \\ &\Rightarrow \quad d(Tg,\, Th) \leq LK \,. \end{aligned}$$

Hence we have that

$$d(Tg,\, Th) \leq L\, d(g,\, h) \,,$$

for all $g, h \in \Omega$, that is, T is a strictly self-mapping of Ω with the Lipschitz constant L . By setting $y = 0$, we have the equation (3.5) as in the proof of Theorem 3.1 and we use the equation (4.3) with the case where $i = 0$, which is reduced to

$$\| f(x) - \frac{1}{8} f(2x) \| \leq \frac{1}{m^2} \frac{1}{2^3} \psi(2x) \leq L\, \psi(x) \,,$$

for all $x \in X$, that is, $d(f, Tf) \leq L = L^1 < \infty$. Now, replacing x by $\frac{1}{2}x$ in the equation (3.5), multiplying 8, and using the equation (4.3) with the case where $i = 1$, we have that

$$\| f(x) - 2^3 f(\frac{x}{2}) \| \leq \psi(x) \,,$$

for all $x \in X$, that is, $d(f, Tf) \leq 1 = L^0 < \infty$. In both cases we can apply the fixed point alternative and since $\lim_{r\to\infty} d(T^r f, C) = 0$, there exists a fixed point C of T in Ω such that

$$C(x) = \lim_{r\to\infty} \frac{f(\lambda_i^r x)}{\lambda_i^{3r}} \,, \tag{4.5}$$

for all $x \in X$. Letting $x = \lambda_i^r x$ and $y = \lambda_i^r y$ in the equation (4.1) and dividing by λ_i^{3r} ,

$$\begin{aligned} \| DC(x,y) \| &= \lim_{r\to\infty} \frac{\| Df(\lambda_i^r x, \lambda_i^r y) \|}{\lambda^{3r}} \\ &\leq \lim_{r\to\infty} \frac{\| \phi(\lambda_i^r x, \lambda_i^r y) \|}{\lambda^{3r}} = 0\,, \end{aligned}$$

for all $x, y \in X$; that is it satisfies the equation (1.3). By Lemma 2.1, the C is cubic. Also, the fixed point alternative guarantees that such a C is the unique function such that

$$\| f(x) - C(x) \| \leq K\,\psi(x)\,,$$

for all $x \in X$ and some $K > 0$. Again using the fixed point alternative, we have

$$d(f,C) \leq \frac{1}{1-L} d(f,Tf)\,.$$

Hence we may conclude that

$$d(f,C) \leq \frac{L^{1-i}}{1-L}\,,$$

which implies the equation (4.4). □

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