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GENERALIZED BIPRODUCT HOPF ALGEBRAS

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Abstract. The smash product algebra has been generalized to general smash product algebra in [3] and we can generalize the smash coproduct coalgebra to obtain the general smash coproduct coalgebra. It is natural to replace the smash product and smash coproduct by the generalized smash product and generalized smash coproduct and consider the condition under which the generalized smash product algebra structure and the generalized smash coproduct coalgebra structure will inherit a bialgebra structure or a Hopf algebra structure. We derive necessary sufficient conditions for the problem. This generalizes the corresponding results in [7] and [4].

1. Preliminary

Let $(H, m_H, \eta_H, \Delta_H, \varepsilon_H)$ be a bialgebra and let (A, m_A, η_A) be an algebra and $(C, \Delta_C, \varepsilon_C)$ be a coalgebra. When we give H-module structure maps $\tau_A : H \otimes A \longrightarrow A,\ h \otimes a \longmapsto h \cdot a$ and $\tau_C : H \otimes C \longrightarrow C,\ h \otimes c \longmapsto h \cdot c$ and H-comodule structure maps $\psi_A : A \longrightarrow H \otimes A,\ a \longmapsto \Sigma a_{-1} \otimes a_0$ and $\psi_C : C \longrightarrow H \otimes C,\ c \longmapsto \Sigma c_{-1} \otimes c_0$, we make the following definitions.

Definition 1. An algebra (A, m_A, η_A) is said to be a *left H-module algebra* if m_A and η_A are left H-module maps and A is a left H-module. That is, if

(i) A is a left H-module, via $\tau_A : H \otimes A \longrightarrow A,\ h \otimes a \longmapsto h \cdot a$

(ii) $\tau_A(h \otimes ab) = \Sigma(h_1 \cdot a)(h_2 \cdot b)$

(iii) $\tau_A(h \otimes 1_A) = \varepsilon(h)1_A \qquad$ for all $h \in H,\ \ a, b \in A$.

Definition 2. An algebra (A, m_A, η_A) is said to be a *left H-comodule algebra* if m_A and η_A are left H-comodule maps and A is a left H-comodule. That is, if

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(i) A is a left H-comodule, via $\psi_A : A \longrightarrow H \otimes A,\ a \longmapsto \Sigma a_{-1} \otimes a_0$

(ii) $\psi_A(1_A) = 1_H \otimes 1_A$

(iii) $\psi_A(ab) = \Sigma a_{-1}b_{-1} \otimes a_0 b_0$ for $a, b \in A$.

DEFINITION 3. A coalgebra $(C, \Delta_C, \varepsilon_C)$ is said to be a *left H-module coalgebra* if Δ_C and ϵ_C are left H-module maps and C is a left H-module. That is, if

(i)C is a left H-module, via $\tau_C : H \otimes C \longrightarrow C,\ h \otimes c \longmapsto h \cdot c$

(ii) $\epsilon_C(\tau_C(h \otimes c)) = \epsilon_H(h)\epsilon_C(c)$

(iii)$\Delta_C(\tau_C(h \otimes c)) = \Sigma h_1 \cdot c_1 \otimes h_2 \cdot c_2$ for $h \in H,\ c \in C$.

DEFINITION 4. A coalgebra $(C, \Delta_C, \varepsilon_C)$ is said to be a *left H-comodule coalgebra* if Δ_C and ϵ_C are left H-comodule maps and C is a left H-comodule. That is, if

(i) C is a left H-comodule, via $\psi_C : C \longrightarrow H \otimes C,\ c \longmapsto \Sigma c_{-1} \otimes c_0$

(ii) $(id \otimes \epsilon_C)(\psi_C(c)) = \eta_H \varepsilon_C(c)$

(iii) $(id \otimes \Delta_C)(\psi_C(c)) = \Sigma(c_1)_{-1}(c_2)_{-1} \otimes (c_1)_0 \otimes (c_2)_0$ for $h \in H,\ c \in C$.

2. Generalized smash product algebras

The "usual" smash product $A\#H$ of an H-module algebra A and a Hopf algebra H has been defined in [1] or [8].

DEFINITION 5. Let H be a bialgebra and A be a left H-module algebra. Then the *smash product algebra* $A\#H$ is defined as follows: for all $a, b \in A,\ h, k \in H$,

(i) as k-spaces, $A\#H = A \otimes H$. We write $a\#h$ for the element $a \otimes h$

(ii) multiplication is given by

$$(a\#h)(b\#k) = \Sigma a(h_1 \cdot b)\#h_2 k,$$

and unit $1_A \otimes 1_H$ for all $a, b \in A$ and $h, k \in H$.

The smash product algebra can be generalized. Takeuchi constructed a smash product of a left H-module algebra A and a left H-comodule algebra D where H is a Hopf algebra [9].

DEFINITION 6. Let H be a bialgebra and A be a left H-module algebra. Let D be a left H-comodule algebra. The *generalized smash product* $A\#_H^L D$

is defined to be $A \otimes_k D$ as a vector space, with multiplication given by

$$(a\#_H^L d)(b\#_H^L e) = \Sigma a(d_{-1} \cdot b)\#_H^L d_0 e$$

and unit $1_A \otimes 1_D$ for all $a, b \in A$ and $d, e \in D$.

It is straightforward to show that $i_A : A \longrightarrow A\#_H^L D,\ a \longmapsto a\#_H^L 1_D$ and $i_D : D \longrightarrow A\#_H^L D,\ d \mapsto 1_A\#_H^L d$ are algebra maps since A is a left H-module algebra and D is a left H-comodule algebra.

EXAMPLE 1. H is a left H-comodule algebra via Δ_H because Δ_H is an algebra map. Moreover, the definition of multiplication in Definition 6 reduces to the multiplication in a smash product, and so $A\#_H^L H = A\#H$.

PROPOSITION 1. *$A\#_H^L D$ is an associative algebra with identity element $1_A\#_H^L 1_D$.*

Proof. Let $m : (A\#_H^L D) \otimes (A\#_H^L D) \longrightarrow (A\#_H^L D)$ be the multiplication in $A\#_H^L D$.

We check that m is associative. Now for $a, b, c \in A,\ \ d, e, f \in D$

$m(m \otimes id)((a\#_H^L d) \otimes (b\#_H^L e) \otimes (c\#_H^L f))$
$= \Sigma m((\Sigma a(d_{-1} \cdot b)\#_H^L d_0 e) \otimes (c\#_H^L f))$
$= \Sigma a(d_{-1} \cdot b)((d_0 e)_{-1} \cdot c)\#_H^L (d_0 e)_0 f$
$= \Sigma a(d_{-1} \cdot b)((d_{0,-1} e_{-1}) \cdot c)\#_H^L ((d_{0,0} e_0) f$ ($\because D$ is a left H-comodule algebra.)
$= \Sigma a(d_{-1} \cdot b)((d_{0,-1} \cdot (e_{-1} \cdot c))\#_H^L (d_{0,0} e_0) f$ ($\because A$ is a left H-module.)
$= \Sigma a[\Sigma((d_{-1,1} \cdot b)((d_{-1,2} \cdot (e_{-1} \cdot c))]\#_H^L d_0(e_0 f)$ ($\because D$ is a left H-comodule.)
$= \Sigma a[d_{-1} \cdot (b(e_{-1} \cdot c))]\#_H^L d_0(e_0 f)$ ($\because A$ is a left H-module algebra.)
$= \Sigma m((a\#_H^L d)) \otimes (\Sigma b(e_{-1} \cdot c)\#_H^L e_0 f)$
$= m(id \otimes m)((a\#_H^L d) \otimes (b\#_H^L e) \otimes (c\#_H^L f))$.

We check that $(1_A\#_H^L 1_D)$ is unit. For any $a \in A,\ \ d \in D$ we have

$(a\#_H^L d)(1_A\#_H^L 1_D) = \Sigma a(d_1 \cdot 1_A)\#_H^L d_0 1_D$
$= \Sigma a\varepsilon(d_{-1})1_A\#_H^L d_0$ ($\because A$ is a left H-module algebra.)
$= \Sigma a\#_H^L \Sigma\varepsilon(d_{-1})d_0$

$$= a\#_H^L d.$$

$$(1_A\#_H^L 1_D)(a\#_H^L d) = \Sigma 1_A\{(1_D)_{-1}\cdot a\}\#_H^L(1_D)_0 d$$

$$= \Sigma 1_A(1_H\cdot a)\#_H^L 1_D d \ \ (\because D \text{ is a left } H\text{-comodule algebra.})$$

$$= a\#_H^L d. \ \ (\because A \text{ is a left } H\text{-module.})$$

This completes the proof. □

3. Generalized smash coproduct coalgebras

In [4], Montgomery introduced smash coproduct of a bialgebra H and a right H-comodule coalgebra C, although we use H instead of H^{cop} since we will be using left and not right comodule algebras. This is in fact a formal dual version of the usual smash product.

Definition 7. Let H be a bialgebra and C be a left H-comodule coalgebra. The *smash coproduct* $C\natural H$ of C and H is a coalgebra described as follows : for all $c\in C,\ h\in H$,

(i) as k-spaces, $C\natural H = C\otimes H$. We write $c\natural h$ for the element $c\otimes h$

(ii) comultiplication is given by

$$\Delta(c\natural h) = \Sigma(c_1\natural c_{2,-1}h_1)\otimes(c_{2,0}\natural h_2),$$

and counit $\varepsilon_{C\natural H}(c\natural h) = \varepsilon_C(c)\otimes\varepsilon_H(h)$.

It is straightforward to show that $\pi_C : C\natural H \longrightarrow C,\ c\natural h \longmapsto c\,\varepsilon_H(h)$ and $\pi_H : C\natural H \longrightarrow C,\ c\natural h \longmapsto \varepsilon_C(c)h$ are coalgebra surjections.

Example 2. Let H be a bialgera and V be a vector space over k. Then V is a trivial left H-comodule by $\psi_V(v) = 1\otimes v$ with the trivial H-comodule structure any coalgebra C over k is an H-comodule coalgebra. Observe that $C\natural H = C\otimes H$ as coalgebras in the trivial case, and that the tensor product structure arise only from the trivial actions.

The smash coproduct algebra can be generalized. Caenepeel constructed a smash coproduct of a right H-module coalgebra C and a right H-comodule coalgebra E where H is a Hopf algebra [2]. Although we use a bialgebra H instead of a Hopf algebra H and we will use a left H-comodule coalgebra C

and a left H-module coalgebra E. Our definition is different from the one in [2].

DEFINITION 8. Let H be a bialgebra and C be a left H-comodule coalgebra. Let E be a left H-module coalgebra. The *generalized smash coproduct* $C\sharp_H^L E$ is defined to be $C \otimes E$ as a vector space with comultiplication given by

$$\Delta_{C\sharp_H^L E}(c\sharp_H^L e) = \Sigma(c_1\sharp_H^L c_{2,-1} \cdot e_1) \otimes (c_{2,0}\sharp_H^L e_2)$$

and counit

$$\varepsilon_{C\sharp_H^L E}(c\sharp_H^L e) = \varepsilon_C(c)\varepsilon_E(e)$$

for all $c \in C$, $e \in E$.

The maps $\pi_C : C\sharp_H^L E \longrightarrow C$, $c\sharp_H^L e \longmapsto \varepsilon_E(e)c$ and $\pi_E : C\sharp_H^L E \longrightarrow E$, $c\sharp_H^L e \longmapsto \varepsilon_C(c)e$ are coalgebra maps, and $(\pi_C \otimes \pi_E)\cdot\Delta = I_{C\sharp_H^L E}$.

EXAMPLE 3. H is a left H-module coalgebra via m_H because m_H is a coalgebra map. Moreover, the definition of comultiplication in Definition 8 reduces to the comultiplication in a smash coproduct, and so $C\sharp_H^L H = C\sharp H$.

PROPOSITION 2. *$(C\sharp_H^L E,\ \Delta_{C\sharp_H^L E},\ \varepsilon_{C\sharp_H^L E})$ is a coalgebra.*

Proof. Since C is a left H-comodule, we have

$$\Sigma c_{-1} \otimes c_{0,-1} \otimes c_{0,0} = \Sigma c_{-1,1} \otimes c_{-1,2} \otimes c_0, \quad c \in C \tag{1}$$

Since C is a left H-comodule coalgebra, we have

$$\Sigma c_{-1} \otimes c_{0,1} \otimes c_{0,2} = \Sigma c_{1,-1} \otimes c_{2,-1} \otimes c_{1,0} \otimes c_{2,0} \tag{2}$$

Replacing c by $c_{2,2}$ in (1), we have

$$\Sigma c_{2,2,-1} \otimes c_{2,2,0,-1} \otimes c_{2,2,0,0} = \Sigma c_{2,2,-1,1} \otimes c_{2,2,-1,2} \otimes c_{2,2,0} \tag{3}$$

Replacing c by c_2 in (2), we have

$$\Sigma c_{2,-1} \otimes c_{2,0,1} \otimes c_{2,0,2} = \Sigma c_{2,1,-1} c_{2,2,-1} \otimes c_{2,1,0} \otimes c_{2,2,0} \tag{4}$$

We check that $\Delta_{C\sharp_H^L E}$ is coassociative. Now for $c \in C$, $e \in E$,

$$\begin{aligned}
&(\Delta_{C\sharp_H^L E} \otimes id)\Delta_{C\sharp_H^L E}(c\sharp_H^L e)\\
&= (\Delta_{C\sharp_H^L E} \otimes id)(\Sigma(c_1\sharp_H^L c_{2,-1} \cdot e_1) \otimes (c_{2,0}\sharp_H^L e_2))\\
&= \Sigma\Delta_{C\sharp_H^L E}(c_1\sharp_H^L c_{2,-1} \cdot e_1) \otimes (c_{2,0}\sharp_H^L e_2)\\
&= \Sigma[(c_{1,1}\sharp_H^L c_{1,2,-1} \cdot (c_{2,-1} \cdot e_1)_1) \otimes (c_{1,2,0}\sharp_H^L (c_{2,-1} \cdot e_1)_2)] \otimes (c_{2,0}\sharp_H^L e_2)
\end{aligned}$$

$= \Sigma[(c_1\sharp_H^L c_{2,1,-1}\cdot(c_{2,2,-1}\cdot e_1)_1)\otimes(c_{2,1,0}\sharp_H^L(c_{2,2,-1}\cdot e_1)_2)]\otimes$
$(c_{2,2,0}\sharp_H^L e_2)\quad(\because C \text{ is a coalgebra.})$
$= \Sigma[(c_1\sharp_H^L c_{2,1,-1}\cdot(c_{2,2,-1,1}\cdot e_{1,1}))\otimes(c_{2,1,0}\sharp_H^L c_{2,2,-1,2}\cdot e_{1,2})]$
$\otimes(c_{2,2,0}\sharp_H^L e_2)\quad(\because E \text{ is a left } H\text{-module coalgebra.})$
$= \Sigma[(c_1\sharp_H^L c_{2,1,-1}\cdot(c_{2,2,-1,1}\cdot e_1))\otimes(c_{2,1,0}\sharp_H^L c_{2,2,-1,2}\cdot e_{2,1})]$
$\otimes(c_{2,2,0}\sharp_H^L e_{2,2})\quad(\because E \text{ is a coalgebra.})$
$= \Sigma(c_1\sharp_H^L(c_{2,1,-1}c_{2,2,-1,1})\cdot e_1)\otimes(c_{2,1,0}\sharp_H^L c_{2,2,-1,2}\cdot e_{2,1})\otimes(c_{2,2,0}\sharp_H^L$
$e_{2,2})\quad(\because E \text{ is a left } H\text{-module.})$
$= \Sigma(c_1\sharp_H^L(c_{2,1,-1}c_{2,2,-1})\cdot e_1)\otimes(c_{2,1,0}\sharp_H^L c_{2,2,0,-1}\cdot e_{2,1})\otimes(c_{2,2,0,0}\sharp_H^L$
$e_{2,2})\quad(\text{by } (3))$
$= \Sigma(c_1\sharp_H^L(c_{2,-1})\cdot e_1)\otimes[(c_{2,0,1}\sharp_H^L c_{2,0,2,-1}\cdot e_{2,1})\otimes(c_{2,0,2,0}\sharp_H^L e_{2,2})]$
$(\text{by } (4))$
$= \Sigma(c_1\sharp_H^L c_{2,-1}\cdot e_1)\otimes\Delta_{C\sharp_H^L E}(c_{2,0}\sharp_H^L e_2)$
$= (id\otimes\Delta_{C\sharp_H^L E})(\Sigma(c_1\sharp_H^L c_{2,-1}\cdot e_1)\otimes(c_{2,0}\sharp_H^L e_2))$
$= (id\otimes\Delta_{C\sharp_H^L E})\Delta_{C\sharp_H^L E}(c\sharp_H^L e).$

We check that $\varepsilon_{C\sharp_H^L E}$ is a counit. Now for $c\in C, e\in E$, we have
$(id\otimes\varepsilon_{C\sharp_H^L E})\Delta_{C\sharp_H^L E}(c\sharp_H^L e) = (id\otimes\varepsilon_{C\sharp_H^L E})(\Sigma(c_1\sharp_H^L c_{2,-1}\cdot e_1)\otimes$
$(c_{2,0}\sharp_H^L e_2))$
$= \Sigma(c_1\sharp_H^L c_{2,-1}\cdot e_1)\otimes\varepsilon_{C\sharp_H^L E}(c_{2,0}\sharp_H^L e_2)$
$= \Sigma(c_1\sharp_H^L c_{2,-1}\cdot e_1)\otimes\varepsilon_C(c_{2,0})\varepsilon_E(e_2)$
$= \Sigma(c_1\sharp_H^L\varepsilon_C(c_{2,0})c_{2,-1}\cdot\varepsilon(e_2)e_1)\otimes 1$
$= \Sigma(c_1\sharp_H^L\varepsilon(c_2)1_H\cdot e)\otimes 1\quad(\because C \text{ is a left } H\text{-comodule coalgebra.})$
$= \Sigma(\varepsilon(c_2)c_1\sharp_H^L e)\otimes 1$
$= (c\sharp_H^L e)\otimes 1.$

A similar computation shows that

$$(\varepsilon_{C\sharp_H^L E}\otimes id)\Delta_{C\sharp_H^L E}(c\sharp_H^L e) = 1\otimes(c\sharp_H^L e).$$

This completes the proof. □

With the smash coproduct in hand, Radford defined the biproduct in [7].

4. Generalized biproduct Hopf algebras

DEFINITION 9. Let H be a bialgebra and let B be a left H-module algebra and a left H-comodule coalgebra. The *biproduct* $B \times H$ of B and H is defined to be $B\#H$ as an algebra and $B\natural H$ as a coalgebra.

With the generalized smash coproduct in hand, we can define the generalized biproduct.

DEFINITION 10. Let H be a bialgebra. Let B be a left H-module algebra and a left H-comodule coalgebra. Let D be a left H-comodule algebra and a left H-module coalgebra. The *generalized biproduct* $B \times_H^L D$ of B and D is defined to be $B\#_H^L D$ as an algebra and $B\natural_H^L D$ as a coalgebra.

EXAMPLE 4. H is a left H-comodule algebra via Δ_H because Δ_H is an algebra map. H is a left H-module coalgebra via m_H because m_H is a coalgebra map. From Example 1 and Example 2, $B \times_H^L H = B \times H$ called the biproduct.

PROPOSITION 3. *Assume that* $\varepsilon_D(1_D) = 1_k$. *Then* $\varepsilon_{B\times_H^L D}$ *is an algebra map if and only if* ε_B *and* ε_D *are algebra maps and* $\Sigma\varepsilon_B(d_{-1}\cdot b)\varepsilon_D(d_0) = \varepsilon_D(d)\varepsilon_B(b)$ *for all* $d \in D$, $b \in B$.

Proof. First, assume that $\varepsilon_{B\times_H^L D}$ is an algebra map. Then

$\varepsilon_{B\times_H^L D}((b \times_H^L d)(b' \times_H^L d')) = \varepsilon_{B\times_H^L D}(b \times_H^L d)\varepsilon_{B\times_H^L D}(b' \times_H^L d')$,

$\varepsilon_{B\times_H^L D}(1_B \times_H^L 1_D) = 1_k$.

By the definition

$\varepsilon_{B\times_H^L D}(1_B \times_H^L 1_D) = \varepsilon_B(1_B)\varepsilon_D(1_D) = 1_k$.

Since $\varepsilon_D(1_D) = 1_k, \varepsilon_B(1_B) = 1_k$.

$\varepsilon_{B\times_H^L D}((b \times_H^L d)(b' \times_H^L d')) = \varepsilon_{B\times_H^L D}(\Sigma b(d_{-1}\cdot b') \times_H^L (d_0 d')) =$
$\Sigma\varepsilon_B(b(d_{-1}\cdot b'))\varepsilon_D(d_0 d')$,

$\varepsilon_{B\times_H^L D}(b \times_H^L d)\varepsilon_{B\times_H^L D}(b' \times_H^L d') = \varepsilon_B(b)\varepsilon_D(d)\varepsilon_B(b')\varepsilon_D(d')$.

Let $d = 1$ and $d' = 1$. Since D is a left H- comodule algebra,

$\Sigma\varepsilon_B(b(1_H\cdot b'))\varepsilon_D(1_D 1_D) = \varepsilon_B(b)\varepsilon_D(1_D)\varepsilon_B(b')\varepsilon_D(1_D)$.

$\varepsilon_B(bb') = \varepsilon_B(b)\varepsilon_B(b')$.

Therefore ε_B is an algebra map. Let $b = 1$ and $b' = 1$. Since B is a left H-module algebra and D is a left H-comodule,

$\Sigma\varepsilon_B(1_B(d_{-1}\cdot 1_B))\varepsilon_D(d_0d') = \Sigma\varepsilon_B(1_B(\varepsilon_H(d_{-1})1_B))\varepsilon_D(d_0d')$
$= \varepsilon_B(1_B)\varepsilon_D((\Sigma\varepsilon_H(d_{-1})d_0)d') = \varepsilon_B(1_B)\varepsilon_D(dd') = \varepsilon_D(dd')$.
$\varepsilon_D(dd') = \varepsilon_B(1_B)\varepsilon_D(d)\varepsilon_B(1_B)\varepsilon_D(d') = \varepsilon_D(d)\varepsilon_D(d')$.

Therefore ε_D is an algebra map.

$\Sigma\varepsilon_B(b)\varepsilon_B(d_{-1}\cdot b')\varepsilon_D(d_0)\varepsilon_D(d') = \varepsilon_B(b)\varepsilon_D(d)\varepsilon_B(b')\varepsilon_D(d')$.

Choose $b \in B,\ d' \in D$ such that $0 \neq \varepsilon_B(b) \in k,\ 0 \neq \varepsilon_D(d') \in k$ because $\varepsilon_B \neq 0$ and $\varepsilon_D \neq 0$. Then

$\Sigma\varepsilon_B(d_{-1}\cdot b')\varepsilon_D(d_0) = \varepsilon_D(d)\varepsilon_B(b')$.

Conversely, assume that ε_B and ε_D are algebra maps and $\Sigma\varepsilon_B(d_{-1}\cdot b)\varepsilon_D(d_0) = \varepsilon_D(d)\varepsilon_B(b)$ holds for $b \in B,\ d \in D$.

$$\begin{aligned}\varepsilon_{B\times^L_H D}((b\times^L_H d)(b'\times^L_H d')) &= \Sigma\varepsilon_B(b(d_{-1}\cdot b'))\varepsilon_D(d_0)\varepsilon_D(d')\\ &= \Sigma\varepsilon_B(b)\varepsilon_B(d_{-1}\cdot b')\varepsilon_D(d_0)\varepsilon_D(d')\\ &= \varepsilon_B(b)\varepsilon_D(d)\varepsilon_B(b')\varepsilon_D(d')\\ &= \varepsilon_{B\times^L_H D}(b\times^L_H d)\varepsilon_{B\times^L_H D}(b'\times^L_H d').\end{aligned}$$

and

$$\varepsilon_{B\times^L_H D}(1_B\times^L_H 1_D) = \varepsilon_B(1_B)\varepsilon_D(1_D) = 1_k\cdot 1_k = 1_k.$$

This completes the proof. □

Corollary 1. *Assume that $\varepsilon_B(h\cdot b) = \varepsilon_H(h)\varepsilon_B(b)$ for all $h \in H, b \in B$ and $\varepsilon_D(1_D) = 1_k$. Then $\varepsilon_{B\times^L_H D}$ is an algebra map if and only if ε_B and ε_D are algebra maps.*

Lemma 1. *Let H be a bialgebra. Let B be a left H-comodule coalgebra. Then*

(i) $\Sigma b_{-1}\otimes b_0 = \Sigma\ \varepsilon_B(b_1)b_{2,-1} \otimes b_{2,0} = \Sigma\ \varepsilon_B(b_2)b_{1,-1} \otimes b_{1,0}$
(ii) $\Sigma b_1\otimes b_2 = \Sigma\ b_1\ \varepsilon_H(b_{2,-1}) \otimes b_{2,0} = \Sigma\ \varepsilon_H(b_{1,-1})b_{1,0} \otimes b_2$
(iii) $\Sigma(\alpha b)_1\otimes(\alpha b)_2 = \Sigma\alpha b_1\otimes b_2$
(iv) $\Sigma(\alpha b)_{-1}\otimes(\alpha b)_0 = \Sigma\alpha b_{-1}\otimes b_0$,

for all $\alpha \in k$ and $b \in B$.

Proof. (i): Since B is a coalgebra, $\psi_B(b) = \psi_B(\Sigma\varepsilon_B(b_1)b_2) = \Sigma\varepsilon_B(b_1)b_{2,-1}\otimes b_{2,0}$ and $\psi_B(b) = \psi_B(\Sigma\varepsilon_B(b_2)b_1) = \Sigma\varepsilon_B(b_2)b_{1,-1}\otimes b_{1,0}$.

(ii): Since B is a left H-comodule , $\Sigma\, \varepsilon_H(b_{2,-1})b_{2,0} = b_2$.

(iii),(iv): Since Δ_B and ψ_B are k-linear maps,

$\Sigma(\alpha b)_1 \otimes (\alpha b)_2 = \Delta_B(\alpha b) = \alpha\Delta_B(b) = \Sigma\alpha b_1 \otimes b_2$,

$\Sigma(\alpha b)_{-1} \otimes (\alpha b)_0 = \psi_B(\alpha b) = \alpha\psi_B(b) = \Sigma\alpha b_{-1} \otimes b_0$. □

Lemma 2. $\Delta_B(b) \cong \Sigma\{b_1 \times^L_H 1_D\} \otimes \{\varepsilon_D(b_{2,-1} \cdot (1_D)_1)b_{2,0} \times^L_H (1_D)_2), b \in B$.

Proof.

$\Sigma(b_1 \times^L_H 1_D) \otimes (\varepsilon_D(b_{2,-1} \cdot (1_D)_1)b_{2,0} \times^L_H (1_D)_2)$

$= \Sigma(b_1 \times^L_H 1_D) \otimes (\varepsilon_H(b_{2,-1})\varepsilon_D((1_D)_1)b_{2,0} \times^L_H (1_D)_2)$ ($\because D$ is a left H-module coalgebra.)

$= \Sigma(b_1 \times^L_H 1_D) \otimes (\varepsilon_H(b_{2,-1})b_{2,0} \times^L_H \varepsilon_D((1_D)_1)(1_D)_2)$

$= \Sigma(b_1 \times^L_H 1_D) \otimes (b_2 \times^L_H 1_D)$ ($\because B$ is a left H-comodule.)

$\cong \Sigma b_1 \otimes b_2$

$= \Delta_B(b)$. □

Proposition 4. *Assume that* $\varepsilon_D(1_D) = 1_k$. *Then* $\Delta_{B\times^L_H D}(1_B \times^L_H 1_D) = (1_B \times^L_H 1_D) \otimes (1_B \times^L_H 1_D)$ *and* $\varepsilon_{B\times^L_H D}(1_B \times^L_H 1_D) = 1_k$ *if and only if* $\psi_B(1_B) = 1_H \otimes 1_B$, $\Delta_B(1_B) = 1_B \otimes 1_B$ *and* $\Delta_D(1_D) = 1_D \otimes 1_D$.

Proof. First, assume that $\Delta_{B\times^L_H D}(1_B \times^L_H 1_D) = (1_B \times^L_H 1_D) \otimes (1_B \times^L_H 1_D)$.

$\Sigma\{(1_B)_1 \times^L_H ((1_B)_{2,-1} \cdot (1_D)_1)\} \otimes \{(1_B)_{2,0} \times^L_H (1_D)_2\} = (1_B \times^L_H 1_D) \otimes (1_B \times^L_H 1_D)$.

By the Lemma 2,

$\Delta_B(1_B) = \Sigma\{(1_B)_1 \times^L_H 1_D\} \otimes \{\varepsilon_D((1_B)_{2,-1} \cdot (1_D)_1)(1_B)_{2,0} \times^L_H (1_D)_2\}$

$= (1_B \times^L_H 1_D) \otimes (\varepsilon_D(1_D)1_B \times^L_H 1_D)$

$= (1_B \times^L_H 1_D) \otimes (1_B \times^L_H 1_D)$ (by the assumption.)

$\cong 1_B \otimes 1_B$.

Since H is a left H-comodule algebra via Δ_H and H is a left H-module coalgebra via m_H, we can replace D by H. By the assumption

$\Sigma\{(1_B)_1 \times^L_H (1_B)_{2,-1}\} \otimes \{(1_B)_{2,0} \times^L_H 1_H\}$

$= (1_B \times^L_H 1_H) \otimes (1_B \times^L_H 1_H)$..................,,,,,,,,,,,,,,,,,,,,,.......(20)

Since $\varepsilon_{B\times^L_H D}(1_B \times^L_H 1_D) = 1_k$ and $\varepsilon_D(1_D) = 1_k$, we have $\varepsilon_B(1_B) = 1_k$.

$\psi_B(1_B) = \Sigma\varepsilon_B((1_B)_1)(1_B)_{2,-1} \otimes (1_B)_{2,0}$ (by Lemma 1.)
$\cong \Sigma\{1_B \times_H^L \varepsilon_B((1_B)_1)(1_B)_{2,-1}\} \otimes \{(1_B)_{2,0} \times_H^L 1_H\}$
$= \Sigma(1_B \times_H^L \varepsilon_B(1_B)1_H\} \otimes \{1_B \times_H^L 1_H\}$ (by 20.)
$= (1_B \times_H^L 1_H) \otimes (1_B \times_H^L 1_H)$
$\cong 1_H \otimes 1_B$.

$\Delta_D(1_D) = \Sigma(1_D)_1 \otimes (1_D)_2$
$\cong \Sigma(1_B \times_H^L (1_D)_1) \otimes (1_B \times_H^L (1_D)_2)$
$= \Sigma\{1_B \times_H^L 1_H \cdot (1_D)_1\} \otimes \{1_B \times_H^L (1_D)_2\}$
$= \Sigma\{1_B \times_H^L ((1_B)_{-1} \cdot (1_D)_1)\} \otimes \{(1_B)_0 \times_H^L (1_D)_2\}$ ($\because \psi_B(1_B) = 1_H \otimes 1_B$.)
$= \Sigma\{(1_B)_1 \times_H^L (1_B)_{2,-1} \cdot (1_D)_1)\} \otimes \{(1_B)_{2,0} \times_H^L (1_D)_2\}$ ($\because \Delta_B(1_B) = 1_B \otimes 1_B$.)
$= \Sigma(1_B \times_H^L 1_D) \otimes (1_B \times_H^L 1_D)$ (by the assumption.)
$\cong 1_D \otimes 1_D$.

Conversely,
$\Sigma\{(1_B)_1 \times_H^L ((1_B)_{2,-1} \cdot (1_D)_1)\} \otimes \{(1_B)_{2,0} \times_H^L (1_D)_2\}$
$= \Sigma\{1_B \times_H^L ((1_B)_{-1} \cdot (1_D)_1)\} \otimes \{(1_B)_0 \times_H^L (1_D)_2\}$ ($\because \Delta_B(1_B) = 1_B \otimes 1_B$.)
$= \Sigma\{1_B \times_H^L (1_H \cdot (1_D)_1\} \otimes \{1_B \times_H^L (1_D)_2\}$ ($\because \psi_B(1_B) = 1_H \otimes 1_B$.)
$= (1_B \times_H^L (1_D)_1) \otimes (1_B \times_H^L (1_D)_2)$
$= (1_B \times_H^L 1_D) \otimes (1_B \times_H^L 1_D)$. ($\because \Delta_D(1_D) = 1_D \otimes 1_D$.)
Therefore $\Delta_{B\times_H^L D}(1_B \times_H^L 1_D) = (1_B \times_H^L 1_D) \otimes (1_B \times_H^L 1_D)$.

Because $\Delta_B(1_B) = 1_B \otimes 1_B$, $1_B = \Sigma\varepsilon_B((1_B)_1)(1_B)_2 = \varepsilon_B(1_B)1_B$.

Since B is a k-vector space, $\varepsilon_B(1_B) = 1_k$. Therefore $\varepsilon_{B\times_H^L D}(1_B \times_H^L 1_D) = 1_k$. □

Lemma 3. *$\Delta_{B\times_H^L D}$ is multiplicative if and only if*

$$(*)\qquad \begin{cases} \Sigma b_1([b_{2,-1} \cdot d_1]_{-1} \cdot b'_1) \times_H^L [b_{2,-1} \cdot d_1]_0[b'_{2,-1} \cdot d'_1] \\ \qquad \otimes b_{2,0}[d_{2,-1} \cdot b'_{2,0}] \times_H^L d_{2,0}d'_2 \\ = \Sigma[b(d_{-1} \cdot b')]_1 \times_H^L [b(d_{-1} \cdot b')]_{2,-1} \cdot (d_{0,1}d'_1) \\ \qquad \otimes[b(d_{-1} \cdot b')]_{2,0} \times_H^L d_{0,2}d'_2 \end{cases}$$

where $b, b' \in B$ *and* $d, d' \in D$.

Proof. From the relation

$\Delta_{B\times^L_H D}((b \times^L_H d)(b' \times^L_H d')) = \Delta_{B\times^L_H D}(b \times^L_H d)\Delta_{B\times^L_H D}(b' \times^L_H d')$

and the direct calculation of

$$\begin{aligned}
&\Delta_{B\times^L_H D}(b \times^L_H d)\Delta_{B\times^L_H D}(b' \times^L_H d')\\
&= (\Sigma b_1 \times^L_H b_{2,-1} \cdot d_1 \otimes b_{2,0} \times^L_H d_2)(\Sigma b'_1 \times^L_H b'_{2,-1} \cdot d'_1 \otimes b'_{2,0} \times^L_H d'_2)\\
&= \Sigma(b_1 \times^L_H [b_{2,-1} \cdot d_1])(b'_1 \times^L_H [b'_{2,-1} \cdot d'_1]) \otimes (b_{2,0} \times^L_H d_2)(b'_{2,0} \times^L_H d'_2)\\
&= \Sigma b_1([b_{2,-1} \cdot d_1]_{-1} \cdot b'_1) \times^L_H [b_{2,-1} \cdot d_1]_0[b'_{2,-1} \cdot d'_1] \otimes b_{2,0}[d_{2,-1} \cdot b'_{2,0}]\\
&\quad \times^L_H d_{2,0}d'_2
\end{aligned}$$

and

$$\begin{aligned}
&\Delta_{B\times^L_H D}((b \times^L_H d)(b' \times^L_H d'))\\
&= \Delta_{B\times^L_H D}(\Sigma b(d_{-1} \cdot b') \times^L_H d_0 d')\\
&= \Sigma[b(d_{-1} \cdot b')]_1 \times^L_H [b(d_{-1} \cdot b')]_{2,-1} \cdot (d_0 d')_1 \otimes [b(d_{-1} \cdot b')]_{2,0} \times^L_H (d_0 d')_2\\
&= \Sigma[b(d_{-1} \cdot b')]_1 \times^L_H [b(d_{-1} \cdot b')]_{2,-1} \cdot d_{0,1}d'_1 \otimes [b(d_{-1} \cdot b')]_{2,0} \times^L_H d_{0,2}d'_2\ ,
\end{aligned}$$

we have the desired result. □

Proposition 5. *Assume that* $\Delta_{B\times^L_H D}$ *is multiplicative. If* ε_B *is an algebra homomorphism and* $\varepsilon_B(h \cdot b) = \varepsilon_H(h)\varepsilon_B(b)$ *for all* $h \in H, b \in B$. *then*

$$\begin{aligned}
&\Sigma 1_B \times^L_H (b_{-1} \cdot d_1)(b'_{-1} \cdot d'_1) \otimes b_0(d_{2,-1} \cdot b'_0) \times^L_H d_{2,0}d'_2\\
&= \Sigma 1_B \times^L_H [b(d_{-1} \cdot b')]_{-1} \cdot (d_{0,1}d'_1) \otimes [b(d_{-1} \cdot b')]_0 \times^L_H d_{0,2}d'_2
\end{aligned}$$

where $b, b' \in B$ *and* $d, d' \in D$.

Proof. Assume that ε_B is an algebra map and $\varepsilon_B(h \cdot b) = \varepsilon_H(h)$ $\varepsilon_B(b)$. We will apply ε_B to the left-hand tensorands of the equation of in Lemma 3 $(*)$.

$$\begin{aligned}
&\Sigma\varepsilon_B(b_1([b_{2,-1} \cdot d_1]_{-1}b'_1))1_B \times^L_H [b_{2,-1} \cdot d_1]_0(b'_{2,-1} \cdot d'_1) \otimes b_{2,0}(d_{2,-1}\cdot\\
&\qquad b'_{2,0}) \times^L_H (d_{2,0}d'_2)\\
&= \Sigma\varepsilon_B(b_1)\varepsilon_H([b_{2,-1} \cdot d_1]_{-1})\varepsilon_B(b'_1)1_B \times^L_H [b_{2,-1} \cdot d_1]_0(b'_{2,-1} \cdot d'_1)\otimes\\
&\qquad b_{2,0}(d_{2,-1} \cdot b'_{2,0}) \times^L_H (d_{2,0}d'_2)
\end{aligned}$$

$= \Sigma 1_B \times_H^L \varepsilon_B(b_1)\varepsilon_H([b_{2,-1} \cdot d_1]_{-1})[b_{2,-1} \cdot d_1]_0 \varepsilon_B(b'_1)(b'_{2,-1} \cdot d'_1) \otimes$
$b_{2,0}(d_{2,-1} \cdot b'_{2,0}) \times_H^L (d_{2,0}d'_2)$
$= \Sigma 1_B \times_H^L \varepsilon_B(b_1)[b_{2,-1} \cdot d_1]\varepsilon_B(b'_1)(b'_{2,-1} \cdot d'_1) \otimes b_{2,0}(d_{2,-1} \cdot b'_{2,0})$
$\times_H^L(d_{2,0}d'_2)$ ($\because$ D is a left H-comodule.)
$= \Sigma 1_B \times_H^L (\varepsilon_B(b_1)b_{2,-1} \cdot d_1)(\varepsilon_B(b'_1)b'_{2,-1} \cdot d'_1) \otimes b_{2,0}(d_{2,-1} \cdot b'_{2,0})$
$\times_H^L(d_{2,0}d'_2)$ (by Lemma 1.)
$= \Sigma 1_B \times_H^L (b_{-1} \cdot d_1)(b'_{-1} \cdot d'_1) \otimes b_0(d_{2,-1} \cdot b'_0)$
$\times_H^L d_{2,0}d'_2$.
$\Sigma \varepsilon_B[b(d_{-1} \cdot b')]_1 1_B \times_H^L [b(d_{-1} \cdot b')]_{2,-1} \cdot (d_{0,1}d'_1) \otimes [b(d_{-1} \cdot b')]_{2,0}$
$\times_H^L d_{0,2}d'_2$
$= \Sigma 1_B \times_H^L [b(d_{-1} \cdot b')]_{-1} \cdot (d_{0,1}d'_1) \otimes [b(d_{-1} \cdot b')]_0 \times_H^L d_{0,2}d'_2$ (by Lemma 1.) □

Proposition 6. *Assume that* $\psi_B(1_B) = 1_H \otimes 1_B$ *and* $\Delta_D(1_D) = 1_D \otimes 1_D$. *The equation in Proposition 5 holds if and only if*

(i) $\Sigma 1_B \times_H^L (bb')_{-1} \cdot d' \otimes (bb')_0 \times_H^L 1_D = \Sigma 1_B \times_H^L (b_{-1} \cdot 1_D)(b'_{-1} \cdot d') \otimes b_0 b'_0 \times_H^L 1_D$.
(ii) $\Sigma 1_B \times_H^L (d_{-1} \cdot b')_{-1} \cdot (d_{0,1}d'_1) \otimes (d_{-1} \cdot b')_0 \times_H^L d_{0,2}d'_2$
$= \Sigma 1_B \times_H^L d_1(b'_{-1} \cdot d'_1) \otimes d_{2,-1} \cdot b'_0 \times_H^L d_{2,0}d'_2$

where $b, b' \in B$ *and* $d, d' \in D$.

Proof. First, note that (i) follows from Proposition 5 by letting d=1. Since D is a left H-comodule algebra,

$\Sigma 1_B \times_H^L (b_{-1} \cdot 1_D)(b'_{-1} \cdot d'_1) \otimes b_0(1_H \cdot b'_0) \times_H^L d'_2$
$= \Sigma 1_B \times_H^L [b(1_H \cdot b')]_{-1} \cdot (d_1)' \otimes [b(1_H \cdot b')]_0 \times_H^L d'_2$.

Since B is a left H-module,

$\Sigma 1_B \times_H^L (b_{-1} \cdot 1_D)(b'_{-1} \cdot d'_1) \otimes b_0 b'_0 \times_H^L d'_2$
$= \Sigma 1_B \times_H^L (bb')_{-1} \cdot d'_1 \otimes (bb')_0 \times_H^L d'_2$.

If apply ε_D to the right-hand tensorands of the above equation,

$\Sigma 1_B \times_H^L (b_{-1} \cdot 1_D)(b'_{-1} \cdot d'_1) \otimes b_0 b'_0 \times_H^L 1_D$
$= \Sigma 1_B \times_H^L (bb')_{-1} \cdot d' \otimes (bb')_0 \times_H^L 1_D$.

Note that (ii) follows from Proposition 5 by letting b=1.

$\Sigma 1_B \times_H^L d_1(b'_{-1} \cdot d'_1) \otimes d_{2,-1} \cdot b'_0 \times_H^L d_{2,0}d'_2$
$= \Sigma 1_B \times_H^L (1_H \cdot d_1)(b'_{-1} \cdot d'_1) \otimes 1_B(d_{2,-1} \cdot b'_0) \times_H^L d_{2,0}d'_2$
$= \Sigma 1_B \times_H^L ((1_B)_{-1} \cdot d_1)(b'_{-1} \cdot d'_1) \otimes (1_B)_0(d_{2,-1} \cdot b'_0) \times_H^L d_{2,0}d'_2$
$= \Sigma 1_B \times_H^L [1_B(d_{-1} \cdot b')]_{-1} \cdot (d_{0,1}d'_1) \otimes [1_B(d_{-1} \cdot b')]_0 \times_H^L d_{0,2}d'_2$(by Proposition 5.)
$= \Sigma 1_B \times_H^L (d_{-1} \cdot b')_{-1} \cdot (d_{0,1}d'_1) \otimes (d_{-1} \cdot b')_0 \times_H^L d_{0,2}d'_2$.

Conversely, assume that (i) and (ii) holds.

$\Sigma 1_B \times_H^L [b(d_{-1} \cdot b')]_{-1} \cdot (d_{0,1}d'_1) \otimes [b(d_{-1} \cdot b')]_0 \times_H^L d_{0,2}d'_2$
$= \Sigma 1_B \times_H^L (b_{-1} \cdot 1_D)[(d_{-1} \cdot b')_{-1} \cdot (d_{0,1}d'_1)] \otimes b_0(d_{-1} \cdot b')_0 \times_H^L d_{0,2}d'_2$ (by (i).)
$= \Sigma 1_B \times_H^L (b_{-1} \cdot 1_D)d_1(b'_{-1} \cdot d'_1) \otimes b_0(d_{2,-1} \cdot b'_0) \times_H^L d_{2,0}d'_2$ (by (ii).)
$= \Sigma 1_B \times_H^L (b_{-1} \cdot 1_D)((1_B)_{-1} \cdot d_1)(b'_{-1} \cdot d'_1) \otimes b_0(1_B)_0(d_{2,-1} \cdot b'_0) \times_H^L d_{2,0}d'_2$
$= \Sigma 1_B \times_H^L ((b(1_B))_{-1} \cdot d_1)(b'_{-1} \cdot d'_1) \otimes (b(1_B))_0(d_{2,-1} \cdot b'_0) \times_H^L d_{2,0}d'_2$ (by (i).)
$= \Sigma 1_B \times_H^L (b_{-1} \cdot d_1)(b'_{-1} \cdot d'_1) \otimes b_0(d_{2,-1} \cdot b'_0) \times_H^L d_{2,0}d'_2$. □

Corollary 2. *Assume that $\Delta_{B\times_H^L}$ is multiplicative. If ε_B is an algebra map, $\varepsilon_B(h \cdot b) = \varepsilon_H(h)\varepsilon_B(b)$for all $h \in H$, $b \in B$, $h \cdot 1_D = \varepsilon_H(h)1_D$, $\psi_B(1_B) = 1_H \otimes 1_B$ and $\Delta_D(1_D) = 1_D \otimes 1_D$ then*

$$\Sigma 1_B \times_H^L (b_{-1} \cdot d) \otimes b_0 \times_H^L 1_D = 1_B \times_H^L d \otimes b \times_H^L 1_D, b \in B, d \in D.$$

Proof. By Proposition 6 (i),

$\Sigma 1_B \times_H^L (bb')_{-1} \cdot d' \otimes (bb')_0 \times_H^L 1_D = \Sigma 1_B \times_H^L (b_{-1} \cdot 1_D)(b'_{-1} \cdot d') \otimes b_0 b'_0 \times_H^L 1_D$.

Let $b' = 1_B$ and $d' = d$. Then

$\Sigma 1_B \times_H^L (b_{-1} \cdot 1_D)d \otimes b_0 \times_H^L 1_D = \Sigma 1_B \times_H^L (b_{-1} \cdot d) \otimes b_0 \times_H^L 1_D$.

Since $h \cdot 1_D = \varepsilon(h)1_D$, $\Sigma 1_B \times_H^L (b_{-1} \cdot 1_D)d \otimes b_0 \times_H^L 1_D = 1_B \times_H^L d \otimes b \times_H^L 1_D$. □

Proposition 7. *Assume that $\Delta_{B\times_H^L D}$ is multiplicative. If Σd_{-1} $\varepsilon_D(d_0) = \varepsilon_D(d)1_H$ and $\varepsilon_D(dd') = \varepsilon_D(d)\varepsilon_D(d')$ for all $d, d' \in D$ then*

$\Sigma[b(d_{-1} \cdot b')]_1 \times_H^L 1_D \otimes [b(d_{-1} \cdot b')]_2 \times_H^L d_0 d' = \Sigma b_1 b'_1 \times_H^L 1_D \otimes b_2(d_{-1} \cdot b'_2) \times_H^L d_0 d'$

where $b, b' \in B$ and $d, d' \in D$.

Proof. Applying ε_D to the second tensorand of the equation in Lemma 3 $(*)$, we compute it by Lemma 1 (i) and (ii).

$\Sigma[b(d_{-1}\cdot b')]_1 \times^L_H \varepsilon_D([b(d_{-1}\cdot b')]_{2,-1}\cdot (d_{0,1}d'_1))1_D \otimes [b(d_{-1}\cdot b')]_{2,0}$
$\times^L_H d_{0,2}d'_2$
$= \Sigma[b(d_{-1}\cdot b')]_1 \times^L_H \varepsilon_H([b(d_{-1}\cdot b')]_{2,-1})\varepsilon_D(d_{0,1})\varepsilon_D(d'_1)1_D \otimes [b(d_{-1}\cdot$
$b')]_{2,0} \times^L_H d_{0,2}d'_2$ ($\because D$ is a left H-module coalgebra.)
$= \Sigma[b(d_{-1}\cdot b')]_1 \times^L_H 1_D \otimes [b(d_{-1}\cdot b')]_2 \times^L_H (d_0)d'$ ($\because B$ is a left H-comodule.)

$\Sigma b_1[(b_{2,-1}\cdot d_1)_{-1}\cdot b'_1] \times^L_H \varepsilon_D[(b_{2,-1}\cdot d_1)_0(b'_{2,-1}\cdot d'_1)]1_D \otimes b_{2,0}[(d_{2,-1}\cdot$
$b'_{2,0})] \times^L_H d_{2,0}d'_2$
$= \Sigma b_1[(b_{2,-1}\cdot d_1)_{-1}\cdot b'_1] \times^L_H \varepsilon_D((b_{2,-1}\cdot d_1)_0)\varepsilon_D(b'_{2,-1}\cdot d'_1)1_D$
$\otimes b_{2,0}[d_{2,-1}\cdot b'_{2,0}] \times^L_H d_{2,0}d'_2$ (by the assumption.)
$= \Sigma b_1[(b_{2,-1}\cdot d_1)_{-1}\cdot b'_1] \times^L_H \varepsilon_D((b_{2,-1}\cdot d_1)_0)\varepsilon_H(b'_{2,-1})\varepsilon_D(d'_1)1_D \otimes$
$b_{2,0}[d_{2,-1}\cdot b'_{2,0}] \times^L_H d_{2,0}d'_2$ ($\because D$ is a left H-module coalgebra.)
$= \Sigma b_1[(b_{2,-1}\cdot d_1)_{-1}\cdot b'_1] \times^L_H \varepsilon_D((b_{2,-1}\cdot d_1)_0)1_D \otimes b_{2,0}[d_{2,-1}\cdot (b'_2)]$
$\times^L_H d_{2,0}d'$ (by Lemma 19, (ii).)
$= \Sigma b_1[\varepsilon(b_{2,-1}\cdot d_1)1_H\cdot b'_1] \times^L_H 1_D \otimes b_{2,0}[d_{2,-1}\cdot b'_2] \times^L_H d_{2,0}d'$ (by the assumption.)
$= \Sigma b_1[\varepsilon(b_{2,-1})\varepsilon(d_1)1_H\cdot b'_1] \times^L_H 1_D \otimes b_{2,0}[d_{2,-1}\cdot b'_2] \times^L_H d_{2,0}d'$ ($\because D$ is a left H-module coalgebra.)
$= \Sigma b_1b'_1 \times^L_H 1_D \otimes b_2(d_{-1}\cdot b'_2) \times^L_H d_0d'$. (by Lemma 1, (i),(ii).) □

Proposition 8. *Assume that $\Delta(1_B) = 1_B \otimes 1_B$. The equation in Proposition 7 holds if and only if*

(i) $\Delta(bb') = \Sigma b_1b'_1 \otimes b_2b'_2$.
(ii) $\Sigma(d_{-1}\cdot b')_1 \times^L_H 1_D \otimes (d_{-1}\cdot b')_2 \times^L_H d_0d' = \Sigma b'_1 \times^L_H 1_D \otimes (d_{-1}\cdot b'_2) \times^L_H d_0d'$.

where $b, b' \in B$ and $d, d' \in D$.

Proof. ($\Rightarrow$) Note that (i) follows from Proposition 6 by letting $d = 1$, $d' = 1$.

$\Sigma[b(1_D)_{-1}\cdot b')]_1 \times^L_H 1_D \otimes [b(1_D)_{-1}\cdot b')]_2 \times^L_H (1_D)_0$
$= \Sigma b_1 b'_1 \times^L_H 1_D \otimes b_2((1_D)_{-1}\cdot b'_2) \times^L_H (1_D)_0.$

Since D is a left H-comodule algebra,

$\Sigma[b(1_H\cdot b')]_1 \times^L_H 1_D \otimes [b(1_H\cdot b')]_2 \times^L_H 1_D = \Sigma b_1 b'_1 \times^L_H 1_D \otimes b_2(1_H\cdot b'_2) \times^L_H 1_D.$

Since B is a left H-module,

$\Sigma(bb')_1 \times^L_H 1_D \otimes (bb')_2 \times^L_H 1_D = \Sigma b_1 b'_1 \times^L_H 1_D \otimes b_2 b'_2 \times^L_H 1_D.$

Therefore

$\Sigma(bb')_1 \otimes (bb')_2 = \Sigma b_1 b'_1 \otimes b_2 b'_2.$

Note that (ii) follows from Proposition 7 by letting $b=1$. Since $\Delta(1_B) = 1_B \otimes 1_B$,

$\Sigma(d_{-1}\cdot b')_1 \times^L_H 1_D \otimes (d_{-1}\cdot b')_2 \times^L_H d_0 d' = \Sigma b'_1 \times^L_H 1_D \otimes (d_{-1}\cdot b'_2) \times^L_H d_0 d'.$

($\Leftarrow$) $\Sigma[b(d_{-1}\cdot b')]_1 \times^L_H 1_D \otimes [b(d_{-1}\cdot b')]_2 \times^L_H d_0 d'$
$= \Sigma b_1 (d_{-1}\cdot b')_1 \times^L_H 1_D \otimes b_2 (d_{-1}\cdot b')_2 \times^L_H d_0 d'$ (by (i).)
$= \Sigma b_1 b'_1 \times^L_H 1_D \otimes b_2 (d_{-1}\cdot b'_2) \times^L_H d_0 d'$ (by (ii).) □

LEMMA 4. *Assume that $\Delta_{B\sharp^L_H D}$ is multiplicative. If ε_B is an algebra map and $\varepsilon_B(h\cdot b) = \varepsilon_H(h)\varepsilon_B(b)$ for all $h \in H$ and $b \in B$ then*

$\Sigma b(d_{1,-1}\cdot b') \times^L_H d_{1,0} d'_1 \otimes 1_B \times^L_H d_2 d'_2 = \Sigma b(d_{-1}\cdot b') \times^L_H d_{0,1} d'_1 \otimes$
$1_B \times^L_H d_{0,2} d'_2.$

Proof. Apply ε_B to the third tensorands of the equation of in Lemma 3 (∗).

$\Sigma b_1([b_{2,-1}\cdot d_1]_{-1}\cdot b'_1) \times^L_H [b_{2,-1}\cdot d_1]_0 (b'_{2,-1}\cdot d'_1) \otimes \varepsilon_B(b_{2,0})\varepsilon_H(d_{2,-1})$
$\varepsilon_B(b'_{2,0}) 1_B \times^L_H d_{2,0} d'_2$
$= \Sigma b_1(\varepsilon_B(b_{2,0})[b_{2,-1}\cdot d_1]_{-1}\cdot b'_1) \times^L_H [b_{2,-1}\cdot d_1]_0 (b'_{2,-1}\cdot d'_1) \otimes \varepsilon_H(d_{2,-1})$
$\varepsilon_B(b'_{2,0}) 1_B \times^L_H d_{2,0} d'_2$
$= \Sigma b_1[\varepsilon_B(b_{2,0})(b_{2,-1}\cdot d_1)]_{-1}\cdot b'_1) \times^L_H [\varepsilon_B(b_{2,0})(b_{2,-1}\cdot d_1)]_0 (b'_{2,-1}\cdot d'_1) \otimes$
$\varepsilon_H(d_{2,-1})\varepsilon_B(b'_{2,0}) 1_B \times^L_H d_{2,0} d'_2$ (by Lemma 1 (iv))
$= \Sigma b_1([\varepsilon_B(b_2) d_1]_{-1}\cdot b'_1) \times^L_H [\varepsilon_B(b_2) d_1]_0 (\varepsilon_B(b'_2) d'_1) \otimes 1_B \times^L_H d_2 d'_2$
($\because$ D is a left H-comodule and B is a left H-comodule coalgebra bra.)
$= \Sigma b_1(\varepsilon_B(b_2) d_{1,-1}\cdot b'_1) \times^L_H d_{1,0}(\varepsilon_B(b'_2) d'_1) \otimes 1_B \times^L_H d_2 d'_2$ (by Lemma

1 (iv))
$= \Sigma\varepsilon_B(b_2)b_1(d_{1,-1}\cdot\varepsilon_B(b'_2)b'_1)\times^L_H d_{1,0}d'_1\otimes 1_B\times^L_H d_2d'_2$
$= \Sigma b(d_{1,-1}\cdot b')\times^L_H d_{1,0}d'_1\otimes 1_B\times^L_H d_2d'_2.$

$\Sigma[b(d_{-1}\cdot b')]_1\times^L_H([b(d_{-1}\cdot b')]_{2,-1})\cdot(d_{0,1}d'_1)\otimes\varepsilon_B([b(d_{-1}\cdot b')]_{2,0})1_B$
$\times^L_H d_{0,2}d'_2$
$= \Sigma[b(d_{-1}\cdot b')]_1\times^L_H\varepsilon_B([b(d_{-1}\cdot b')]_2)1_H\cdot(d_{0,1}d'_1)\otimes 1_B\times^L_H d_{0,2}d'_2$
($\because B$ is a left H-comodule coalgebra.)
$= \Sigma b(d_{-1}\cdot b')\times^L_H d_{0,1}d'_1\otimes 1_B\times^L_H d_{0,2}d'_2.$ □

Corollary 3. *Assume that* $\Delta_{B\sharp^L_H D}$ *is multiplicative. Suppose that* ε_B *is an algebra map,* $\varepsilon_B(h\cdot b)=\varepsilon_H(h)\varepsilon_B(b)$, $\Sigma d_{-1}\varepsilon_D(d_0)=\varepsilon_D(d)1_H$ *and* $\varepsilon_D(dd')=\varepsilon_D(d)\varepsilon_D(d')$ *for all* $h\in H$, $b\in B$ $d,d'\in D$. *Then*

$$\Sigma d_{-1}\cdot b'\times^L_H 1_D\otimes 1_B\times^L_H d_0 = b'\times^L_H 1_D\otimes 1_B\times^L_H d.$$

Proposition 9. *Assume that* $\Delta_{B\sharp^L_H D}$ *is multiplicative. Suppose that* ε_B *is an algebra map,* $\varepsilon_B(h\cdot b)=\varepsilon_H(h)\varepsilon_B(b)$, $h\cdot 1_D=\varepsilon_H(h)1_D$, $\psi_B(1_B)=1_H\otimes 1_B$, $\Delta_D(1_D)=1_D\otimes 1_D$, $\Sigma d_{-1}\varepsilon_D(d_0)=\varepsilon_D(d)1_H$, $\varepsilon_D(dd')=\varepsilon_D(d)\varepsilon_D(d')$ *for all* $h\in H$, $b\in B$ $d,d'\in D$. *Then*

$$\Sigma b'\times^L_H(b_{-1}\cdot d)\otimes b_0\times^L_H 1_D = \Sigma(b_{-1}\cdot d)_{-1}\cdot b'\times^L_H(b_{-1}\cdot d)_0\otimes b_0\times^L_H 1_D$$

for all $b,b'\in B$, $d\in D$.

Proof. $\Sigma(b_{-1}\cdot d)_{-1}\cdot b'\times^L_H(b_{-1}\cdot d)_0\otimes b_0\times^L_H 1_D$
$= \Sigma d_{-1}\cdot b'\times^L_H d_0\otimes b\times^L_H 1_D$ (by Corollary 2)
$= b'\times^L_H d\otimes b\times^L_H 1_D$ (by Corollary 3)
$= \Sigma b'\times^L_H(b_{-1}\cdot d)\otimes b_0\times^L_H 1_D$ (by Corollary 2.) □

Theorem 1. *Let* H *be a bialgebra over a filed* k, *and suppose* B *is an algebra in* ${}_HM$ *and a coalgebra in* ${}^HM$ *such that* $\varepsilon_B(h\cdot b)=\varepsilon_H(h)\varepsilon_B(b)$ *for all* $b\in B$, $h\in H$. *Suppose* D *is an algebra in* ${}^HM$ *and a coalgebra in* ${}_HM$ *such that* $h\cdot 1_D=\varepsilon_H(h)1_D$, $\varepsilon_D(1_D)=1_k$ *and* $\Sigma d_{-1}\varepsilon_D(d_0)=\varepsilon_D(d)1_H$ *for all* $d\in D$, $h\in H$. *Then the followings are equivalent ;*

(a) $(B\times^L_H D, m_{B\#^L_H D}, \eta_{B\#^L_H D}, \Delta_{B\sharp^L_H D}, \varepsilon_{B\sharp^L_H D})$ *is a bialgebra.*

(b) ε_B *and* ε_D *are algebra maps,* $\Delta_B(1_B) = 1_B \otimes 1_B$*,* $\Delta_D(1_D) = 1_D \otimes 1_D, \psi_B(1_B) = 1_H \otimes 1_B$*, and the identities*

(i) $\Sigma 1_B \times_H^L (b_{-1} \cdot d_1)(b'_{-1} \cdot d'_1) \otimes b_0(d_{2,-1} \cdot b'_0) \times_H^L d_{2,0}d'_2$
$= \Sigma 1_B \times_H^L [b(d_{-1} \cdot b')]_{-1} \cdot (d_{0,1}d'_1) \otimes [b(d_{-1} \cdot b')]_0 \times_H^L d_{0,2}d'_2.$

(ii) $\Sigma[b(d_{-1} \cdot b')]_1 \times_H^L 1_D \otimes [b(d_{-1} \cdot b')]_2 \times_H^L d_0 d' = \Sigma b_1 b'_1 \times_H^L 1_D \otimes b_2(d_{-1} \cdot b'_2) \times_H^L d_0 d'$

(iii) $\Sigma b' \times_H^L (b_{-1} \cdot d) \otimes b_0 \times_H^L 1_D = \Sigma (b_{-1} \cdot d)_{-1} \cdot b' \times_H^L (b_{-1} \cdot d)_0 \otimes b_0 \times_H^L 1_D$

hold for $b, b' \in B$ *and* $d, d' \in D$*.*

Proof. (a)⇒(b). ε_B and ε_D are algebra maps by Corollary 1 and $\Delta_B(1_B) = 1_B \otimes 1_B$, $\Delta_D(1_D) = 1_D \otimes 1_D$ and $\psi_B(1_B) = 1_H \otimes 1_B$ by Proposition 4. The equality (i) follows from Proposition 5, the equality (ii)follows from Proposition 7 and the equality (iii)follows from Proposition 9.

(b)⇒(a). By Corollary 1, $\varepsilon_{B\sharp_H^L D}$ is an algebra map, and by Proposition 4 to show that $\Delta_{B\sharp_H^L D}$ is an algebra map we need only from that $\Delta_{B\sharp_H^L D}$ is multiplicative. But for this it suffices to show by Lemma 3 that,

$$\Sigma[b(d_{-1} \cdot b')]_1 \times ([b(d_{-1} \cdot b')]_{2,-1}) \cdot (d_{0,1}d'_1) \otimes ([b(d_{-1} \cdot b')]_{2,0}) \times d_{0,2}d'_2$$
$$= \Sigma b_1 b'_1 \times [b_2(d_{-1} \cdot b'_2)]_{-1} \cdot d_{0,1}d'_1 \otimes [b_2(d_{-1} \cdot b'_2)]_0 \times d_{0,2}d'_2 \quad \text{(by (ii) where } d' = 1_D)$$
$$= \Sigma b_1 b'_1 \times [b_{2,-1} \cdot d_1][b'_{2,-1} \cdot d'_1] \otimes b_{2,0}[d_{2,-1} \cdot b'_{2,0}] \times d_{2,0}d'_2 \quad \text{(by (i))}$$
$$= \Sigma b_1([b_{2,-1} \cdot d_1]_{-1} \cdot b'_1) \times [b_{2,-1} \cdot d_1]_0[b'_{2,-1} \cdot d'_1] \otimes b_{2,0}[d_{2,-1} \cdot b'_{2,0}] \times d_{2,0}d'_2 \quad \text{(by (iii))}.$$
□

THEOREM 2. *Let* H, B *and* D *be as Theorem 1. Then the followings are equivalent:*

(a) $(B \times D, m_{B\#_H^L D}, \eta_{B\#_H^L D}, \Delta_{B\sharp_H^L D}, \varepsilon_{B\sharp_H^L D})$ *is a bialgebra.*

(b) ε_B *and* ε_D *are algebra maps,* $\Delta_B(1_B) = 1_B \otimes 1_B$*,* $\Delta_D(1_D) = 1_D \otimes 1_D, \psi_B(1_B) = 1_H \otimes 1_B$ *, and the following identities hold :*

(i) $\Sigma 1_B \times_H^L (bb')_{-1} \cdot d' \otimes (bb')_0 \times_H^L 1_D = \Sigma 1_B \times_H^L (b_{-1} \cdot 1_D)(b'_{-1} \cdot d') \otimes b_0 b'_0 \times_H^L 1_D$

(ii) $\Sigma 1_B \times_H^L (d_{-1} \cdot b')_{-1} \cdot (d_{0,1}d'_1) \otimes (d_{-1} \cdot b')_0 \times_H^L d_{0,2}d'_2 = \Sigma d_1(b'_{-1} \cdot$

$d'_1) \otimes d_{2,-1} \cdot b'_0 \times d_{2,0}d'_2$

(iii) $\Delta_B(bb') = \Sigma b_1 b'_1 \otimes b_2 b'_2$

(iv) $\Sigma(d_{-1} \cdot b')_1 \times^L_H 1_D \otimes (d_{-1} \cdot b')_2 \times^L_H d_0 d' = \Sigma b'_1 \times^L_H 1_D \otimes (d_{-1} \cdot b'_2) \times^L_H d_0 d'$.

Proof. It follows from Proposition 6, Theorem 1 and Proposition 8. □

DEFINITION 11. In case $(B \times^L_H D, m_{B\#^L_H D}, \eta_{B\#^L_H D}, \Delta_{B\sharp^L_H D}, \varepsilon_{B\sharp^L_H D}$) is a bialgebra we say the pair (D, B) is *admissible.*

THEOREM 3. *Suppose that (D, B) is admissible. If B has an antipode S_B and D has an antipode S_D then $B \times^L_H D$ is a Hopf algebra with antipode given by*

$$S(b \times^L_H d) = \Sigma(1_B \times^L_H S_D(b_{-1} \cdot d))(S_B(b_0) \times^L_H 1_B).$$

Proof.

$\Sigma(S(b \times^L_H d)_1)(b \times^L_H d)_2$

$= \Sigma(S(b_1 \times^L_H b_{2,-1} \cdot d_1))(b_{2,0} \times^L_H d_2)$

$= \Sigma[\Sigma\{1_B \times^L_H S_D(b_{1,-1} \cdot (b_{2,-1} \cdot d_1))\}\{S_B(b_{1,0} \times^L_H 1_D\}](b_{2,0} \times^L_H d_2)$

$= \Sigma[\{1_B \times^L_H S_D((b_{1,-1}b_{2,-1} \cdot d_1)\}\{S_B(b_{1,0} \times^L_H 1_D\}](b_{2,0} \times^L_H d_2)$

$= \Sigma[\{1_B \times^L_H S_D(b_{-1} \cdot d_1)\}\{S_B(b_{0,1} \times^L_H 1_D\}](b_{0,2} \times^L_H d_2)$ ($\because$ B is a left H-comodule coalgebra.)

$= \Sigma\{1_B \times^L_H S_D(b_{-1} \cdot d_1)\}[(S_B(b_{0,1} \times^L_H 1_D)(b_{0,2} \times^L_H d_2)]$

$= \Sigma\{1_B \times^L_H S_D(b_{-1} \cdot d_1)\}\{\Sigma S_B(b_{0,1})((1_D)_{-1} \cdot b_{0,2}) \times^L_H (1_D)_0 d_2\}$ (by definition)

$= \Sigma\{1_B \times^L_H S_D(b_{-1} \cdot d_1)\}\{S_B(b_{0,1})(1_H \cdot b_{0,2}) \times^L_H 1_D d_2\}$ ($\because$ D is a left H-comodule algebra.)

$= \Sigma\{1_B \times^L_H S_D(b_{-1} \cdot d_1)\}\{S_B(b_{0,1})b_{0,2} \times^L_H d_2\}$.

$= \Sigma\{1_B \times^L_H S_D(b_{-1} \cdot d_1)\}\{\varepsilon_B(b_0)1_B \times^L_H d_2\}$ ($\because$ S_D is an antipode.)

$= \Sigma\{1_B \times^L_H S_D(\varepsilon_B(b_0)b_{-1} \cdot d_1)\}\{1_B \times^L_H d_2\}$

$= \Sigma\{1_B \times^L_H S_D(\varepsilon_B(b)1_H \cdot d_1)\}\{1_B \times^L_H d_2\}$ ($\because$ B is a left H-comodule coalgebra.)

$= \Sigma\varepsilon_B(b)\{1_B \times^L_H S_D(1_H \cdot d_1)\}\{1_B \times^L_H d_2\}$

$= \Sigma\varepsilon_B(b)(1_B \times_H^L S_D(d_1))(1_B \times_H^L d_2)$ ($\because D$ is a left H-module.)
$= \Sigma\varepsilon_B(b)[\Sigma 1_B\{(S_D(d_1)_{-1} \cdot 1_B\} \times_H^L (S_D(d_1))_0 d_2]$ (by definition.)
$= \Sigma\varepsilon_B(b)\{1_B\varepsilon_H((S_D(d_1))_{-1}1_B \times_H^L (S_D(d_1))_0 d_2\}$ ($\because B$ is a left H-module algebra.)
$= \Sigma\varepsilon_B(b)\{1_B \times_H^L \varepsilon_H((S_D(d_1))_{-1})(S_D(d_1))_0 d_2\}$
$= \Sigma\varepsilon_B(b)(1_B \times_H^L S_D(d_1)d_2)$ ($\because D$ is a left H-comodule.)
$= \varepsilon_B(b)(1_B \times_H^L \varepsilon_D(d)1_D)$
$= \varepsilon_B(b)\varepsilon_D(d)(1_B \times_H^L 1_D)$
$= \varepsilon(b \times_H^L d)1_{B\times_H^L D}$.

Therefore $\Sigma(S(b \times_H^L d)_1)(b \times_H^L d)_2 = \varepsilon(b \times_H^L d)1_{B\times_H^L D}$.

$\Sigma(b \times_H^L d)_1 S((b \times_H^L d)_2)$
$= \Sigma(b_1 \times_H^L b_{2,-1} \cdot d_1)S(b_{2,0} \times_H^L d_2)$
$= \Sigma(b_1 \times_H^L b_{2,-1} \cdot d_1)[\Sigma\{1_B \times_H^L (b_{2,0,-1} \cdot d_2)\}\{S_B(b_{2,0,0}) \times_H^L 1_D\}]$
$= \Sigma(b_1 \times_H^L b_{2,-1,1} \cdot d_1)[\{1_B \times_H^L S_D(b_{2,-1,2} \cdot d_2)\}\{S_B(b_{2,0}) \times_H^L 1_D\}]$ ($\because B$ is a left H-comodule.)
$= \Sigma[(b_1 \times_H^L b_{2,-1,1} \cdot d_1)(1_B \times_H^L S_D(b_{2,-1,2} \cdot d_2))](S_B(b_{2,0}) \times_H^L 1_D)$
$= \Sigma[b_1((b_{2,-1,1} \cdot d_1)_{-1} \cdot 1_B) \times_H^L (b_{2,-1,1} \cdot d_1)_0 S_D(b_{2,-1,2} \cdot d_2)](S_B(b_{2,0}) \times_H^L 1_D)$ (by definition)
$= \Sigma[b_1\varepsilon_H((b_{2,-1,1} \cdot d_1))1_B \times_H^L (b_{2,-1,1} \cdot d_1)_0 S_D(b_{2,-1,2} \cdot d_2)](S_B(b_{2,0}) \times_H^L 1_D)$ ($\because B$ is a left H-module algebra.)
$= \Sigma[b_1 \times_H^L \varepsilon_H((b_{2,-1,1} \cdot d_1)_{-1})(b_{2,-1,1} \cdot d_1)_0 S_D(b_{2,-1,2} \cdot d_2)](S_B(b_{2,0}) \times_H^L 1_D)$
$= \Sigma[b_1 \times_H^L (b_{2,-1,1} \cdot d_1)S_D(b_{2,-1,2} \cdot d_2)](S_B(b_{2,0}) \times_H^L 1_D)$ ($\because D$ is a left H-comodule.)
$= \Sigma[b_1 \times_H^L (b_{2,-1} \cdot d)_1 S_D((b_{2,-1} \cdot d)_2)](S_B(b_{2,0}) \times_H^L 1_D)$ ($\because D$ is a left H-module coalgebra.)
$= \Sigma(b_1 \times_H^L \varepsilon_D(b_{2,-1} \cdot d)1_D)(S_B(b_{2,0}) \times_H^L 1_D)$ ($\because S_D$ is an antipode.)
$= \Sigma(b_1 \times_H^L \varepsilon_H(b_{2,-1})\varepsilon_D(d)1_D)(S_B(b_{2,0}) \times_H^L 1_D)$ ($\because D$ is a left H-module coalgebra.)
$= \Sigma\varepsilon_D(d)(b_1 \times_H^L 1_D)(S_B(\varepsilon_H(b_{2,-1})b_{2,0}) \times_H^L 1_D)$
$= \Sigma\varepsilon_D(d)(b_1 \times_H^L 1_D)(S_B(b_2) \times_H^L 1_D)$ ($\because B$ is a left H-comodule.)

$= \Sigma\varepsilon_D(d)(\Sigma b_1((1_D)_{-1} \cdot S_B(b_2)) \times_H^L (1_D)_0 1_D)$ (by definition.)
$= \Sigma\varepsilon_D(d)(b_1(1_H \cdot S_B(b_2)) \times_H^L 1_D 1_D)$
$= \Sigma\varepsilon_D(d)(b_1 S_B(b_2) \times_H^L 1_D)$ ($\because B$ is a left H-module.)
$= \varepsilon_D(d)(\varepsilon_b(b) 1_B \times_H^L 1_D)$ ($\because S_B$ is an antipode.)
$= \varepsilon_B(b)\varepsilon_D(d)(1_B \times_H^L 1_D)$
$= \varepsilon(b \times_H^L d) 1_{B \times_H^L D}$.
Therefore $\Sigma(b \times_H^L d)_1 S((b \times_H^L d)_2) = \varepsilon(b \times_H^L d) 1_{B \times_H^L D}$. □

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